WHITNEY POLYGONS, SYMBOL HOMOLOGY AND COBORDISM MAPS

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ABSTRACT. We define a new homology theory we call symbol homology by using decorated moduli spaces of Whitney polygons. By decorating different types of moduli spaces we obtain different flavors of this homology theory together with morphisms between them. Each of these flavors encodes the properties of a different type of Heegaard Floer homology. The morphisms between the symbol homologies enable us to push properties from one Floer theory to a different one. Furthermore, we obtain a new presentation of Heegaard Floer theory in which maps correspond to multiplication from the right with suitable elements of our symbol homology. Finally, we present the construction of cobordism maps in knot Floer theories and apply the tools from symbol homology to give an invariance proof.

1. INTRODUCTION

In [4], Ozsváth and Szabó assign to a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ the homology theories $\widehat{\mathrm{HF}}$, HF^- and HF^∞ by studying holomorphic disks in the symmetric product $\mathrm{Sym}^g(\Sigma)$. These homologies turn out to be invariants of the 3-manifold associated to the Heegaard diagram. By altering the construction process, different flavors of this theory were introduced like for instance knot Floer homology or sutured Floer homology. Each of these theories are based on studying moduli spaces of Whitney polygons. The 0-dimensional moduli spaces are used to define maps, and the boundaries of the 1-dimensional moduli spaces provide relations/properties of these maps and of the theory. We call this the *Floer theoretic scheme*, which is common among all existing Floer theories. More precisely, almost all properties of maps in Floer theory are proved by applying the moduli space machinery, i.e. deriving a statement on the moduli space level, and then interpreting this statement on the Floer chain level by counting components of moduli spaces. This observation suggests that Floer theory takes place in an algebraic object formed by moduli spaces themselves, but which is hidden by the procedure of counting elements. The main goal of this paper is to construct such an algebraic object.

We will decorate moduli spaces of Whitney polygons with data. More precisely, we classify three types of decorations and attach these decorations to the vertices of the polygons (see Definition 3.1 or cf. §5). These decorated spaces will be used to generate an algebraic object $(\widehat{\mathcal{T}}, \boxplus, \boxtimes)$ in which we identify a substructure $(\mathcal{S}, \boxplus, \boxtimes)$ we call **symbol algebra** (see Definition 3.8). The connection between the symbol algebra and the Floer theoretic level is a *morphism* $\mathfrak{ev}: (\mathcal{S}, \boxplus, \boxtimes) \longrightarrow (\mathbb{MOR}, +, \circ)$ where $\mathbb{MOR}$ should be thought of as the set of maps in Floer homology (see Proposition 3.13 or cf. §5). In this way, every map in Floer homology which is defined by counting 0-dimensional components of moduli spaces

of Whitney polygons can be described as an element in the symbol algebra. Moreover, in this algebraic setting the codimension-1 boundaries of moduli spaces of Whitney polygons can be described as elements in $\mathcal{S}$: so, we obtain a differential $\partial_{\mathfrak{sh}} \colon \mathcal{S} \longrightarrow \mathcal{S}$ on the symbol algebra. We denote by $\mathfrak{sh}_*$ the homology theory associated to $(\mathcal{S}, \partial_{\mathfrak{sh}})$ and call it **symbol homology**. The morphism $\mathfrak{cv}$ vanishes on boundaries and, hence, a given map f on the Floer theoretic level can be represented by an element $\mathfrak{s}$ in the symbol homology in the sense that $\mathfrak{cv}(\mathfrak{s}) = f$. By the Floer theoretic scheme mentioned above, a property of a map f – which is proved by applying the moduli space machinery – can be encoded into a polynomial expression P with coefficients in $\mathfrak{sh}_*$ in such a way that f fulfills the property if $P(\mathfrak{s}) = 0$ (cf. Example 5.3). Hence, the symbol homology captures the information provided by the Floer theory and, therefore, seems to be a reasonable candidate for the algebraic object mentioned in the first paragraph.

The first construction of a symbol homology is given in §3 and done using moduli spaces which are relevant for the $\widehat{\mathrm{HF}}$-theory.
A particular appealing feature of this theory is that we obtain a nice interpretation of Heegaard Floer theory in terms of our algebraic setting. In fact, given a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ we find a module $\mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ in $\widehat{\mathcal{T}}$ which is naturally equipped with a differential $\partial_{\mathcal{X}}$, such that

$$(1.1) \qquad (\mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2, \partial_{\mathcal{X}} \otimes \mathrm{id}) \cong (\widehat{\mathrm{CF}}(\mathcal{H}), \widehat{\partial}_{\mathcal{H}})$$

as chain complexes (see Theorem 6.1). Furthermore, suppose we are given another Heegaard diagram $\mathcal{H}'$ and a map f between the Heegaard Floer chain modules $\widehat{\mathrm{CF}}(\mathcal{H})$ and $\widehat{\mathrm{CF}}(\mathcal{H}')$, where we denote by $\mathfrak{s}_f$ the symbol which represents f, then multiplication from the right with this symbol defines a map $\cdot \boxtimes \mathfrak{s}_f \boxtimes \mathbb{O} \colon \mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \longrightarrow \mathcal{X}^0_{(\boldsymbol{\alpha}', \boldsymbol{\beta}')}$ such that the following diagram commutes (see Theorem 6.1)

$$(1.2)$$

$$
\begin{array}{ccc}
\mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 & \xrightarrow{\ \cong\ } & \widehat{\mathrm{CF}}(\mathcal{H}) \\
{\scriptstyle (\cdot \boxtimes \mathfrak{s}_f \boxtimes \mathbb{O}) \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle f} \\
\mathcal{X}^0_{(\boldsymbol{\alpha}', \boldsymbol{\beta}')} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 & \xrightarrow{\ \cong\ } & \widehat{\mathrm{CF}}(\mathcal{H}'),
\end{array}
$$

where $\mathbb{O}$ is an element in the symbol homology which just depends on the pair of attaching circles $(\boldsymbol{\alpha}', \boldsymbol{\beta}')$.

1.1. **Applications.** The first construction of a symbol homology is given in §3 by decorating moduli spaces that are relevant for the $\widehat{\mathrm{HF}}$-homology. The construction does not depend on a particular setup or on a particular type of moduli spaces and thus can be done with moduli spaces relevant for other Heegaard Floer homologies. These homologies are then also described in terms which are analogous to (1.1) and (1.2). We outline the construction for moduli spaces relevant to the $\widehat{\mathrm{HFK}}$-homology (and the $\mathrm{HFK}^{\bullet,-}$-homology). To fix notation, we will denote by $\mathcal{S}^w$ the symbol algebra which is generated using moduli spaces relevant for the $\widehat{\mathrm{HFK}}$-homology and denote by $\mathfrak{sh}^w_*$ the associated symbol homology. Recall from above that a property of a map f in the $\widehat{\mathrm{HF}}$-theory can be encoded into a polynomial

expression P with coefficients in $\mathfrak{sh}_*$ such that if $\mathfrak{s}$ is a symbol representing f, then $P(\mathfrak{s}) = 0$ implies that f has the property. So, we say that f has property P or $\mathfrak{s}$ has property P. Defining symbol algebras for different flavors of Heegaard Floer theory naturally give rise to morphisms between them. In this particular case, we define a morphism $\mathfrak{F} \colon \mathcal{S} \longrightarrow \mathcal{S}^w$ the so-called **filtering morphism**. We will see that this morphism is a chain map and, thus, descends to a map $\mathfrak{F}_*$ between the associated symbol homology theories. With this morphism at hand we are able to prove the following statement (cf. also Example 5.4).

Theorem 1.1. *If a map $\mathfrak{s}$ from a tensor product of Heegaard Floer chain complexes to another Heegaard Floer chain complex fulfills a property P, then the filtered map $\mathfrak{F}_*(\mathfrak{s})$ between the corresponding $\widehat{\mathrm{CFK}}$-knot Floer chain complexes fulfills the filtered property $P_{\mathfrak{F}}$.*

We point the reader to §4.2 for a precise definition of $P_{\mathfrak{F}}$. A similar statement can be formulated and proved for a relation between the $\widehat{\mathrm{HF}}$-theory and $\mathrm{HFK}^{\bullet,-}$ (see Theorem 7.2). One of the benefits of the symbol homology theory is that it unifies the Floer chain level and the moduli space level into one object (see discussion in §5, cf. §6 and §3.3). A consequence of this unification is that it provides a systematic and immediate way to transfer properties between different Floer theoretic settings without difficulty. This is indicated by Theorem 1.1 (see also Theorem 7.2). Proofs of properties which need the moduli space machinery now do not need to be repeated in different settings but can now just be accepted by pointing to the results of this paper. This allows a systematic transfer of properties between different flavors of Heegaard Floer theory and can in principle be done for other Floer homologies using the techniques from this paper. To demonstrate how the transfer is set up when explicitly worked out, we give a construction of cobordism maps for knot Floer homologies and apply the techniques from symbol homologies in the proof of the following theorem.

Theorem 1.2. *For a* Spin^c*-structure* $\mathfrak{s}$ *over* W *we have that*

$$F_{W;\mathfrak{s}}^{\bullet,\circ} \colon \mathrm{HFK}^{\bullet,\circ}(Y, K; \mathfrak{s}|_Y) \longrightarrow \mathrm{HFK}^{\bullet,\circ}(Y', K'; \mathfrak{s}|_{Y'})$$

is uniquely defined up to sign.

For a precise definition of all notions and the maps we point the reader to §9. For a definition of $\mathrm{HFK}^{\bullet,\circ}$ we point the reader to §2.2. In fact, we also give further demonstrations in Example 5.4, Corollary 10.1 and in the surgery exact triangle given in Theorem 10.2 which is a generalization of [6, Theorem 8.2] (cf. also [8, Theorem 2.7]).

1.2. **Organization.** The best method to read this article is probably to combine a linear reading with jumps into §5. In §5 we present a somewhat informal introduction to some of the ideas which will help the reader to familiarize with the objects. In fact, we introduce some notational conventions for the decorations in that section which make the whole construction intuitive. Additionally, we present two calculations of symbol homologies in easy situations (see Example 5.1 and Example 5.2).
In §3.1 we give the construction of a symbol homology modeled on the $\widehat{\mathrm{HF}}$-theory. In §3.3 we provide the operation $\mathfrak{ev}$ that connects the symbol algebra with the maps between Floer

chain modules. In §4 we give the construction of the filtered symbol algebra, provide the filtering morphism in §4.1 and describe in §4.2 the implications of the filtering morphism. In §6 we give the new presentation of Floer theory in terms of our symbol homology introduced in §3.1. What is done in that section can potentially be done with every other flavor of symbol homology. In §7 we define another flavor of symbol homology capturing the theory $\mathrm{HFK}^{\bullet,-}$. This should serve as a model for how to introduce a U-variable into the theory. In §8 we outline how to bring moduli spaces with dynamic boundary conditions into the theory. This is necessary to apply the symbol homologies in the proof of Theorem 1.2, since the invariances in Heegaard Floer theory mainly come from morphisms that are defined by considering polygons with dynamic boundary conditions. In §9 we provide the construction of cobordism maps in knot Floer theory, specifically focusing on $\widehat{\mathrm{HFK}}$ and $\mathrm{HFK}^{\bullet,-}$. What is done there can also be applied to all other knot Floer theories with slight adaptions (cf. also §9.5). Finally, in §10 we present a surgery exact triangle for maps induced by knot cobordisms.

Acknowledgments. We wish to thank Kai Cieliebak for helpful conversations.

2. Preliminaries

2.1. **Heegaard Floer homologies.** In Heegaard Floer theory one assigns to a closed, oriented 3-manifold the homology groups $\widehat{\mathrm{HF}}(Y)$, $\mathrm{HF}^-(Y)$, $\mathrm{HF}^\infty(Y)$ and $\mathrm{HF}^+(Y)$ (see [4]). In the following, we will give a brief review of the two versions $\widehat{\mathrm{HF}}(Y)$ and $\mathrm{HF}^-(Y)$.

A Heegaard diagram $\mathcal{H}$ is a triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, where Σ is an oriented genus-g surface and $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\}$, $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ are two sets of pairwise disjoint, simple, closed curves in Σ called **attaching circles**. Both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are required to consist of curves which represent linearly independent classes in $H_1(\Sigma, \mathbb{Z})$. In the following we will talk about the curves in the set $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$) as $\boldsymbol{\alpha}$-**curves** (resp. $\boldsymbol{\beta}$-**curves**). Without loss of generality, we may assume that the $\boldsymbol{\alpha}$-curves and $\boldsymbol{\beta}$-curves intersect transversely.

Now suppose we are given a closed, oriented 3-manifold Y. To Y we associate a Heegaard diagram $\mathcal{H}$ and use it to construct a homology theory as follows: To the Heegaard diagram $\mathcal{H}$ we associate the triple $(\mathrm{Sym}^g(\Sigma), \mathbb{T}_{\boldsymbol{\alpha}}, \mathbb{T}_{\boldsymbol{\beta}})$ consisting of the g-fold symmetric power

$$\mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/S_g$$

of Σ, the submanifold $\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \cdots \times \alpha_g$ of $\mathrm{Sym}^g(\Sigma)$ and the submanifold $\mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \cdots \times \beta_g$ of $\mathrm{Sym}^g(\Sigma)$. We define $\mathrm{CF}^-(\mathcal{H})$ as the $\mathbb{Z}_2[U]$-module generated by the set $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. A map $\phi\colon D^2 \longrightarrow \mathrm{Sym}^g(\Sigma)$ $(D^2 \subset \mathbb{C})$ is called **Whitney disk** if $\phi(D^2 \cap \{Re < 0\}) \subset \mathbb{T}_{\boldsymbol{\alpha}}$ and $\phi(D^2 \cap \{Re > 0\}) \subset \mathbb{T}_{\boldsymbol{\beta}}$. We call $D^2 \cap \{Re < 0\}$ the $\boldsymbol{\alpha}$-**boundary of** ϕ and $D^2 \cap \{Re > 0\}$ the $\boldsymbol{\beta}$-**boundary of** ϕ. For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ we say that a Whitney disk **connects** $\boldsymbol{x}$ **with** $\boldsymbol{y}$ if $\phi(i) = \boldsymbol{x}$ and $\phi(-i) = \boldsymbol{y}$. Denote by $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ the set of homology classes of Whitney disks that connect $\boldsymbol{x}$ with $\boldsymbol{y}$. Note that $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ can be interpreted as a subgroup of $H_2(\mathrm{Sym}^g(\Sigma), \mathbb{T}_{\boldsymbol{\alpha}} \cup \mathbb{T}_{\boldsymbol{\beta}})$. We endow $\mathrm{Sym}^g(\Sigma)$ with a symplectic structure ω. By choosing a path of almost complex structures $\mathcal{J}_s$ on $\mathrm{Sym}^g(\Sigma)$ suitably (cf. [4]), all moduli spaces of holomorphic Whitney disks are Gromov-compact manifolds. We fix a point $z \in \Sigma \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$

and use it to define the map $n_z \colon \pi_2(\boldsymbol{x}, \boldsymbol{y}) \longrightarrow \mathbb{Z}$ which assigns to a Whitney disk ϕ its intersection number with the submanifold $V_z = \{z\} \times \operatorname{Sym}^{g-1}(\Sigma)$. In fact, the path of almost complex structures $\mathcal{J}_s$ is chosen in such a way that V_z is a complex submanifold of the symmetric product. A path of almost complex structures for which V_z is a complex submanifold is called z-**respectful**. For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ denote by $\widehat{\mathcal{M}}^i_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(\boldsymbol{x}, \boldsymbol{y})$ the set of unparametrized holomorphic Whitney disks ϕ which connect $\boldsymbol{x}$ with $\boldsymbol{y}$ such that $n_z(\phi) = i$. For a point $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ we define

$$\partial_{\mathcal{H}}^- \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}, i \geq 0} \#\left(\widehat{\mathcal{M}}^i_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(\boldsymbol{x}, \boldsymbol{y})\right) \cdot U^i \boldsymbol{y}$$

and we extend $\partial_{\mathcal{H}}^-$ to $\mathrm{CF}^-(\mathcal{H})$ as a morphism of $\mathbb{Z}_2[U]$-modules. The map $\partial_{\mathcal{H}}^-$ is a differential. The associated homology theory $H_*(\mathrm{CF}^-(\mathcal{H}), \partial_{\mathcal{H}}^-)$ is denoted by $\mathrm{HF}^-(Y)$ and it is a topological invariant of Y. By setting $U = 0$, we obtain a different flavor of this theory: The associated chain module is denoted by $\widehat{\mathrm{CF}}(\mathcal{H})$ and it can be interpreted as the $\mathbb{Z}_2$-module generated by $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. Restricting $\partial_{\mathcal{H}}^-$ to the module $\widehat{\mathrm{CF}}(\mathcal{H})$, we obtain a map $\widehat{\partial}_{\mathcal{H}} \colon \widehat{\mathrm{CF}}(\mathcal{H}) \longrightarrow \widehat{\mathrm{CF}}(\mathcal{H})$, which is still a differential. In fact, for every $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, the equality

$$\widehat{\partial}_{\mathcal{H}} \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \#\left(\widehat{\mathcal{M}}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(\boldsymbol{x}, \boldsymbol{y})\right) \cdot \boldsymbol{y}$$

holds. The homology theory $H_*(\widehat{\mathrm{CF}}(\mathcal{H}), \widehat{\partial}_{\mathcal{H}})$ will be denoted by $\widehat{\mathrm{HF}}(Y)$. We would like to note that not all Heegaard diagrams are suitable for defining the Heegaard Floer homology groups. There is an additional condition that has to be imposed called *admissibility*. A detailed knowledge of this condition is not important in the remainder of the present article since all constructions are done nicely so that there will never be a problem. We advise the interested reader to [4] .

2.2. **Knot Floer Homology.** Given a knot $K \subset Y$, we can specify a certain subclass of Heegaard diagrams.

Definition 2.1. A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is said to be **adapted** to the knot K if K is isotopic to a knot lying in Σ and K intersects β_1 once transversely and is disjoint from the other $\boldsymbol{\beta}$-curves.

Every pair (Y, K) admits a Heegaard diagram adapted to K. Having fixed such a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, we can encode the knot K in a pair of points: After isotoping K onto Σ, we fix a small interval I in K containing the intersection point $K \cap \beta_1$. This interval is chosen small enough such that I does not contain any other intersections of K with other attaching curves. The boundary ∂I lies in the complement of the attaching circles and consists of two points we denote by z and w such that $\partial I = z - w$ as chains. Here, the orientation of I is given by the knot orientation. In this way, we associate to the pair (Y, K) a doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$.
Conversely, given a doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, we denote by Y the manifold represented by the underlying Heegaard diagram. We connect w with z with an

arc δ in $\Sigma\backslash(\boldsymbol{\alpha}\cup\boldsymbol{\beta}\backslash\beta_1)$ that crosses β_1, once. Then, we connect z with w in $\Sigma\backslash\boldsymbol{\beta}$ using an arc γ. The union $\delta\cup\gamma$ is a knot K we equip with the orientation such that $\partial\delta = z - w$. Hence, we obtain a pair (Y,K).

Suppose we are given a doubly-pointed Heegaard diagram $\mathcal{H} = (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$. The knot chain module $\mathrm{CFK}^{\bullet,-}(\mathcal{H})$ is the free $\mathbb{Z}_2[U]$-module generated by the intersection points $\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$. Analogous to the definitions given above, we define $\widehat{\mathcal{M}}^{(i,j)}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y})$ as the set of unparametrized holomorphic Whitney disks ϕ that connect $\boldsymbol{x}$ with $\boldsymbol{y}$ such that $(n_z(\phi),n_w(\phi))$ equals (i,j). For $\boldsymbol{x}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}}$ we define

$$\partial^{\bullet,-}_{\mathcal{H}}\boldsymbol{x} = \sum_{\boldsymbol{y}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}},j\geq 0} \#\left(\widehat{\mathcal{M}}^{(0,j)}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y})\right)\cdot U^j\boldsymbol{y}.$$

We extend $\partial^{\bullet,-}_{\mathcal{H}}$ to $\mathrm{CFK}^{\bullet,-}(\mathcal{H})$ as a morphism of $\mathbb{Z}_2[U]$-modules. The associated homology theory $H_*(\mathrm{CFK}^{\bullet,-}(\mathcal{H}),\partial^{\bullet,-}_{\mathcal{H}})$ is denoted by $\mathrm{HFK}^{\bullet,-}(Y,K)$ and is an invariant of the pair (Y,K). By setting $U=0$ as before, we obtain a new theory which we denote by $\mathrm{HFK}^{\bullet,\bullet}(Y,K)$ or, alternatively, $\widehat{\mathrm{HFK}}(Y,K)$. It is also possible to define variants such as $\mathrm{HFK}^{\bullet,+}$ and $\mathrm{HFK}^{\bullet,\infty}$. For details we point the reader to [8].

To justify our notation, observe, that it is possible to *swap the roles* of z and w by defining a differential $\partial^{-,\bullet}_{\mathcal{H}}$ via

$$\partial^{-,\bullet}_{\mathcal{H}}\boldsymbol{x} = \sum_{\boldsymbol{y}\in\mathbb{T}_{\boldsymbol{\alpha}}\cap\mathbb{T}_{\boldsymbol{\beta}},j\geq 0} \#\left(\widehat{\mathcal{M}}^{(j,0)}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y})\right)\cdot U^j\boldsymbol{y}.$$

The associated homology theory is denoted by $\mathrm{HFK}^{-,\bullet}(Y,K)$. As in the previous case, we can also define the variants $\mathrm{HFK}^{\infty,\bullet}$ and $\mathrm{HFK}^{+,\bullet}$.

3. The Symbol Homology Package

3.1. **Whitney Polygons and Symbol Homology.** Let Σ be a surface of genus g and $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n$ sets of attaching circles on this surface. A map

$$\phi\colon D^2\longrightarrow \mathrm{Sym}^g(\Sigma)$$

with $p_i\in\partial D^2$, $i=1,\ldots,n$ with boundary conditions in $\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n$ is called **Whitney polygon** of degree n. We may think of D^2 itself as a polygon with vertices $p_1,\ldots,p_n$. For $B=\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n\}$ we consider the disjoint union

$$\bigsqcup_{i=2,\ldots,|B|} B^{\times i}\backslash\Delta_i$$

where Δ_i is the diagonal in $B^{\times i}$. Every element $a\in B^{\times i}\backslash\Delta_i$ specifies boundary conditions on the edges of the polygon ϕ by the following algorithm: For $i=1,\ldots,n-1$ the i-th component of a, $\boldsymbol{\alpha}$ say, specifies that the edge of ϕ between the vertices p_i and p_{i+1} has to be mapped into $\mathbb{T}_{\boldsymbol{\alpha}}$. Analogous, the n-th component of a specifies the boundary condition of the edge between p_n and p_1.

We say that two elements a, b of $B^{\times i}\backslash\Delta_i$ are equivalent if the boundary conditions they

specify can be identified by a rotation of the polygon. This defines an equivalence relation $\sim$ and we denote by $\mathcal{I}_B$ the set of equivalence classes

$$\Big(\bigsqcup_{i=2,\ldots,|B|} B^{\times i} \backslash \Delta_i \Big) / \sim \; .$$

We call $\mathcal{I}_B$ the **index set** or the **set of boundary conditions**. For $B' \in \mathcal{I}_B$ we denote by $\pi_2(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n; B')$ the set of homotopy classes of Whitney polygons connecting the $\boldsymbol{x}_i$ with boundary conditions given by B'. We denote by $\mathcal{M}_{B'}^\mu(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l)$ the space of $\mathcal{J}_s$-holomorphic Whitney polygons that connect the $\boldsymbol{x}_i$ with boundary conditions given by B' and Maslov-index μ.

The set B' specifies boundary conditions on the edges of every polygon $\phi \in \mathcal{M}_{B'}^\mu$. We can additionally impose conditions on the vertices of ϕ which is usually indicated by attaching points into the notation like for instance $\mathcal{M}_{B'}^\mu(\ldots, \boldsymbol{q}, \ldots)$. If conditions on the vertices are specified we call the space **pointed**. Suppose we are given a pointed moduli space $\mathcal{M}_{B'}^\mu(\ldots, \boldsymbol{q}, \ldots)$ of n-gons. Each vertex of the polygons $\phi \in \mathcal{M}_{B'}^\mu$ can be specified by a pair (a_i, a_{i+1}) of attaching circles a_i, a_{i+1} (from B') by the following algorithm: The element B' specifies boundary conditions on the Whitney polygons as described above. If we think of the polygon as sitting in $\mathbb{R}^2$, then the standard orientation on $\mathbb{R}^2$ and, hence, of the polygon induces an orientation on the boundary, i.e. a preferred direction of travel along the boundary. The vertex specified by (a_i, a_{i+1}) is the vertex sitting between the edges to which the boundary conditions a_i and a_{i+1} have been attached, such that traversing through the boundary of the polygon in the preferred direction of travel, we will pass the vertex by approaching from a_i and moving over the vertex to a_{i+1}. By abuse of notation, we will call the pair (a_i, a_{i+1}) **vertex**. A set $\{(a_i, a_{i+1}), \boldsymbol{q}\}$ consisting of a vertex (a_i, a_{i+1}) and a point $\boldsymbol{q} \in \mathbb{T}_{a_i} \cap \mathbb{T}_{a_{i+1}}$ is called a **pointing**.

Definition 3.1. Let $\mathcal{A}$ be the symbol $A_{(P^\bullet, F^\downarrow, F^\uparrow)}$ consisting of the following data:

(1) We have that A represents an unparametrized moduli space $\mathcal{M}_{B'}^\mu$ where B' equals $[(a_1, \ldots, a_n)]$ which is an element of $\mathcal{I}_B$. Furthermore, we require that $\mu \leq 2$ if $n = 2$ and $\mu \leq 1$, otherwise.

(2) The sets $P^\bullet$ and $F^\uparrow$ consist of a collection of pointings for A. We call $P^\bullet$ the **set of pointings** and $F^\uparrow$ the set of **flow-out vertices**.

(3) The set $F^\downarrow$ is called the set of **flow-in vertices** and consists of a collection of vertices of A.

We require that each vertex of A appears in the set $P^\bullet$, in the set $F^\downarrow$ and in the set $F^\uparrow$ at most once. If each vertex of A is either a pointed, a flow-in or a flow-out vertex, then we call $\mathcal{A}$ a **pre-generator**. A pre-generator is called a **generator** if $\#(F^\downarrow) > 0$ and $\#(F^\uparrow) = 1$. A pre-generator with no flow-in vertices and no flow-out vertices is called **fully pointed**.

For a pre-generator $\mathcal{A} = A_{(P^\bullet, F^\downarrow, F^\uparrow)}$ we define $\pi(\mathcal{A})$ to be the moduli space we obtain from A after attaching the slot points given in $P^\bullet$ and $F^\uparrow$ as boundary conditions to the vertices of the polygons in A. Now consider the commutative free polynomial $\mathbb{Z}_2$-algebra generated by fully pointed pre-generators, where we denote the sum by $+$ and the product by $\bullet$. For

each pre-generator $\mathcal{A}$ with the property that $\pi(\mathcal{A}) = \emptyset$ we introduce the relation $\mathcal{A} = 0$. Furthermore, we introduce the relation $\mathcal{A} = 1$ if $\pi(\mathcal{A}) \neq \emptyset$ or if the elements of A are bigons with Maslov-index 0. The algebra we obtain after introducing these relations will be denoted by $\mathbb{F}$ and called the **coefficient algebra**. Then, we consider the free non-commutative $\mathbb{F}$-algebra generated by the set of pre-generators which are not fully pointed, using the disjoint union $+$ as the sum and the Cartesian product $\times$ as the product operation. Denote by $\mathcal{T}$ the algebra we obtain after introducing the relation $\mathcal{A} = 0$ for every pre-generator $\mathcal{A}$ for which $\pi(\mathcal{A})$ is empty. We denote by $\widehat{\mathcal{T}}$ the set $\mathcal{T} \cup \{\widehat{\Omega}\}$.

Definition 3.2. Denote by $\mathcal{R}$ a ring.

(a) A set P is a $\mathcal{R}$-**semimodule** if P is a semigroup together with a map

$$\bullet \colon \mathcal{R} \times P \longrightarrow P$$

such that for all $v, w \in P$ and $a, b \in \mathcal{R}$ we have that $1_{\mathcal{R}} \bullet v = v$, $a \bullet (v+w) = a \bullet v + a \bullet w$, $a \bullet (b \bullet v) = (ab) \bullet v$ and $(a+b) \bullet v = a \bullet v + b \bullet v$.

(b) A set $(P, +)$ is a $\mathcal{R}$-**semialgebra** if P is a $\mathcal{R}$-semimodule with a multiplication, $\times$ say, which is associative and distribute with respect to $+$ and such that for $a, b \in \mathcal{R}$ and $v, w \in P$ we have that $(a \bullet v) \times (b \bullet w)$ equals $ab \bullet (v \times w)$.

A $\mathcal{R}$-semialgebra will also be called **semialgebra** if the ring $\mathcal{R}$ is specified in the context.

It will be our goal to define a $\mathbb{F}$-semialgebra structure on $\widehat{\mathcal{T}}$. We will do this in the following, but we need to provide a couple of definitions beforehand so that we will be able to give clean definitions. The semialgebra structure will be modeled on the disjoint union and the Cartesian product, but the they will also include modifying the decorations (cf. §5).

Definition 3.3. Suppose we are given a pre-generator $\mathcal{A} = A_{(P^{\bullet}, F^{\downarrow}, F^{\uparrow})}$.

(a) For a set Q consisting of pointings (like for instance $F^{\uparrow}$), we define $\mathfrak{D}(Q)$ as the set we obtain from Q by forgetting the slot points.

(b) Given a set Q consisting of pointings, we define the operator $\mathcal{K}^{\downarrow, Q}$ as follows:

$$\mathcal{K}^{\downarrow, Q}(P^{\bullet}, F^{\downarrow}, F^{\uparrow}) = (P^{\bullet} \cup Q, F^{\downarrow} \backslash \mathfrak{D}(Q), F^{\uparrow}).$$

(c) We define the operator $\mathcal{K}^{\uparrow}$ as follows:

$$\mathcal{K}^{\uparrow}(P^{\bullet}, F^{\downarrow}, F^{\uparrow}) = (P^{\bullet} \cup F^{\uparrow}, F^{\downarrow}, \emptyset).$$

(d) Given another pre-generator $\mathcal{B} = B_{(P^{\bullet\prime}, F^{\downarrow\prime} F^{\uparrow\prime})}$, we call the vertices in $\mathfrak{D}(F^{\uparrow}) \cap F^{\downarrow\prime}$ **the common vertices of $\mathcal{A}$ and $\mathcal{B}$.**

Correspondingly, we define $\mathcal{K}^{\downarrow, Q}(\mathcal{A}) = A_{\mathcal{K}^{\downarrow, Q}(P^{\bullet}, F^{\downarrow}, F^{\uparrow})}$ and $\mathcal{K}^{\uparrow}(\mathcal{A}) = A_{\mathcal{K}^{\uparrow}(P^{\bullet}, F^{\downarrow}, F^{\uparrow})}$.

Let $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ be a product of pre-generators. Given a vertex s, we define

$$\mathfrak{m}^{\mathcal{C}}(s) = \#\Big\{ q \in \bigsqcup_{i=1}^{k} F^{\downarrow}(\mathcal{C}_i) \,\big|\, q = s \Big\}$$

which we call the **inward multiplicity of** $\mathcal{C}$ **at** s. Correspondingly, we define

$$\mathfrak{m}_{\mathcal{C}}(s) = \#\Big\{ q \in \bigsqcup_{i=1}^{k} \mathfrak{D}(F^{\uparrow}(\mathcal{C}_i)) \,|\, q = s \Big\}$$

and call it the **outward multiplicity of** $\mathcal{C}$ **at** s.

Lemma 3.4. *For a product of pre-generators* $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ *we have that*

$$\mathfrak{m}^{\mathcal{C}}(s) = \sum_{i=1}^{k} \mathfrak{m}^{\mathcal{C}_i}(s) \ \text{ and } \ \mathfrak{m}_{\mathcal{C}}(s) = \sum_{i=1}^{k} \mathfrak{m}_{\mathcal{C}_i}(s).$$

Proof. This readily follows from the definition of the inward multiplicities and the outward multiplicities. □

Now we will define a sum on $\widehat{\mathcal{T}}$: First, we set $f \bullet \mathcal{A} \boxplus \widehat{\Omega} = f \bullet \widehat{\Omega} \boxplus \mathcal{A} = \widehat{\Omega}$ for all $\mathcal{A} \in \widehat{\mathcal{T}}$ and all $f \in \mathbb{F}$. Second, for given products of pre-generators $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$, $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_l$ and elements $f, g \in \mathbb{F}$ we define

$$f \bullet \mathcal{R} \boxplus g \bullet \mathcal{C} = \begin{cases} f \bullet \mathcal{R} + g \bullet \mathcal{C} & \text{, if } \mathfrak{m}^{\mathcal{C}} \equiv \mathfrak{m}^{\mathcal{R}} \text{ and } \mathfrak{m}_{\mathcal{C}} \equiv \mathfrak{m}_{\mathcal{R}} \\ \widehat{\Omega} & \text{, otherwise} \end{cases}.$$

Finally, for elements $\sum_{i=1}^{k} f_i \bullet q_i$ and $\sum_{j=1}^{l} g_j \bullet r_j$ where the q_i and r_j are defined as products of pre-generators and the f_i and g_j are elements of $\mathbb{F}$, we have that

$$\Big(\sum_{i=1}^{k} f_i \bullet q_i \Big) \boxplus \Big(\sum_{j=1}^{l} g_j \bullet r_j \Big) = \Big(\boxplus_{i=1,\dots,k} f_i \bullet q_i \Big) \boxplus \Big(\boxplus_{j=1,\dots,l} g_j \bullet r_j \Big)$$

These definitions uniquely define a map

$$\boxplus \colon \widehat{\mathcal{T}} \times \widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{T}}.$$

As we see from the definition, whether a $\boxplus$-sum of elements equals $\widehat{\Omega}$ or not depends on the decorations of the polygons. If the $\boxplus$-sum does not equal $\widehat{\Omega}$, we say that **the data of the summands match**.

Proposition 3.5. *The map* $\boxplus$ *is commutative and associative, i.e.* $(\widehat{\mathcal{T}}, \boxplus)$ *is a commutative semigroup.*

Proof. Commutativity of $\boxplus$ readily follows from its definition. To prove associativity, suppose we are given elements

$$a = \sum_{i=1}^{k} f_i \bullet q_i, \ \ b = \sum_{j=1}^{l} g_j \bullet r_j, \ \text{ and } \ c = \sum_{n=1}^{m} h_n \bullet s_n$$

where the q_i, r_j and s_n are suitable products of pre-generators and the f_i, g_j and h_n suitable elements in the coefficient algebra $\mathbb{F}$. Then, $(a \boxplus b) \boxplus c$ equals $\widehat{\Omega}$ unless all the inward multiplicities of the q_i, r_j and s_n agree and all outward multiplicities of the q_i, r_j

and s_n agree. But this is equivalent to the statement that $a \boxplus (b \boxplus c)$ does not equal $\widehat{\Omega}$. Hence, if the data match in the specified sense, we have that

$$(a \boxplus b) \boxplus c = (a + b) + c = a + (b + c) = a \boxplus (b \boxplus c)$$

which completes the proof. $\qquad\square$

The definition of a product on the set $\widehat{\mathcal{T}}$ is slightly more complicated. We point the reader to §5 for a discussion of the idea behind the product. As before, for all $\mathcal{A} \in \widehat{\mathcal{T}}$ and $f \in \mathbb{F}$ we define $(f \bullet \mathcal{A}) \boxtimes \widehat{\Omega} = \widehat{\Omega} \boxtimes (f \bullet \mathcal{A}) = \widehat{\Omega}$. For pre-generators $\mathcal{A}$ and $\mathcal{B}$ with $\mathfrak{D}(F^{\uparrow}(\mathcal{A})) \subset F^{\downarrow\prime}(\mathcal{B})$ and elements $f, g \in \mathbb{F}$, we have that

$$(3.1) \qquad (f \bullet \mathcal{A}) \boxtimes (g \bullet \mathcal{B}) = \begin{cases} \left(f \bullet \mathcal{K}^{\uparrow}(\mathcal{A})\right) \times \left(g \bullet \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B})\right), & \text{if } \mathcal{K}^{\uparrow}(\mathcal{A}), \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B}) \notin \mathbb{F} \\ \left(f \bullet \mathcal{K}^{\uparrow}(\mathcal{A})\right) \bullet \left(g \bullet \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B})\right), & \text{if } \mathcal{K}^{\uparrow}(\mathcal{A}) \in \mathbb{F} \\ \left(f \bullet \mathcal{K}^{\uparrow}(\mathcal{A})\right) \bullet \left(g \bullet \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B})\right), & \text{if } \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B}) \in \mathbb{F} \end{cases}$$

and if $\mathfrak{D}(F^{\uparrow}(\mathcal{A})) \not\subset F^{\downarrow\prime}(\mathcal{B})$, then we set $(f \bullet \mathcal{A}) \boxtimes (g \bullet \mathcal{B}) = \widehat{\Omega}$. Now, given products of pre-generators $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ and $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_l$ and elements $f, g \in \mathbb{F}$, we require that the product $(f \bullet \mathcal{R}) \boxtimes (g \bullet \mathcal{C})$ equals

$$(fg) \bullet \mathcal{R}_1 \times \cdots \times R_{l-1} \bullet (\mathcal{R}_l \boxtimes \mathcal{C}_1) \bullet \mathcal{C}_2 \times \cdots \times \mathcal{C}_k$$

if $\mathcal{K}^{\uparrow}(\mathcal{R}_l)$ and $\mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{C}_1)$ are in $\mathbb{F}$ and we require that it equals

$$(fg) \bullet \mathcal{R}_1 \times \cdots \times \mathcal{R}_{l-1} \times (\mathcal{R}_l \boxtimes \mathcal{C}_1) \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_k,$$

otherwise. Finally, given elements $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in $\widehat{\mathcal{T}}$ we set

$$\begin{aligned} \mathcal{A} \boxtimes (\mathcal{B} + \mathcal{C}) &= (\mathcal{A} \boxtimes \mathcal{B}) \boxplus (\mathcal{A} \boxtimes \mathcal{C}) \\ (\mathcal{B} + \mathcal{C}) \boxtimes \mathcal{A} &= (\mathcal{B} \boxtimes \mathcal{A}) \boxplus (\mathcal{C} \boxtimes \mathcal{A}). \end{aligned}$$

These definitions provide a map

$$\boxtimes \colon \widehat{\mathcal{T}} \times \widehat{\mathcal{T}} \longrightarrow \widehat{\mathcal{T}}.$$

From the definition we see that whether a $\boxtimes$-product of elements equals $\widehat{\Omega}$ or not depends on the decorations of the polygons. If the $\boxtimes$-product of elements does not equal $\widehat{\Omega}$, we say that **the data of the factors match**, or simply that **the data match**.

Proposition 3.6. *The map $\boxtimes$ is associative and $\boxplus$-bilinear, i.e. the equalities*

$$\begin{aligned} \mathcal{P} \boxtimes \left(f \bullet \mathcal{R} \boxplus g \bullet \mathcal{C}\right) &= f \bullet (\mathcal{P} \boxtimes \mathcal{R}) \boxplus g \bullet (\mathcal{P} \boxtimes \mathcal{C}) \\ \left(f \bullet \mathcal{R} \boxplus g \bullet \mathcal{C}\right) \boxtimes \mathcal{P} &= f \bullet (\mathcal{R} \boxtimes \mathcal{P}) \boxplus g \bullet (\mathcal{C} \boxtimes \mathcal{P}) \end{aligned}$$

hold for $\mathcal{P}$, $\mathcal{R}$, $\mathcal{C} \in \widehat{\mathcal{T}}$ and $f, g \in \mathbb{F}$. Hence, the triple $(\widehat{\mathcal{T}}, \boxplus, \boxtimes)$ is a $\mathbb{F}$-semialgebra.

Proof. For pre-generators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ the following equalities hold:

$$\begin{aligned} (\mathcal{A} \times \mathcal{B}) \boxtimes \mathcal{C} &= \mathcal{A} \times (\mathcal{B} \boxtimes \mathcal{C}) \\ \mathcal{A} \boxtimes (\mathcal{B} \times \mathcal{C}) &= (\mathcal{A} \boxtimes \mathcal{B}) \times \mathcal{C} \end{aligned}$$

Hence, if all data match and all factors that appear in (3.2) are not fully pointed, we have the following chain of equalities

$$
\begin{aligned}
(\mathcal{A} \boxtimes \mathcal{B}) \boxtimes \mathcal{C} &= \left(\mathcal{K}^{\uparrow}(\mathcal{A}) \times \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B}) \right) \boxtimes \mathcal{C} \\
&= \mathcal{K}^{\uparrow}(\mathcal{A}) \times \left(\mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B}) \boxtimes \mathcal{C} \right) \\
&= \mathcal{K}^{\uparrow}(\mathcal{A}) \times \mathcal{K}^{\uparrow}(\mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B})) \times \mathcal{K}^{\downarrow, F^{\uparrow\prime}}(\mathcal{C}) \\
&= \mathcal{K}^{\uparrow}(\mathcal{A}) \times \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{K}^{\uparrow}(\mathcal{B})) \times \mathcal{K}^{\downarrow, F^{\uparrow\prime}}(\mathcal{C}) \\
&= \mathcal{A} \boxtimes \left(\mathcal{K}^{\uparrow}(\mathcal{B}) \times \mathcal{K}^{\downarrow, F^{\uparrow\prime}}(\mathcal{C}) \right) \\
&= \mathcal{A} \boxtimes \left(\mathcal{B} \boxtimes \mathcal{C} \right)
\end{aligned}
\tag{3.2}
$$

where the fourth equality holds since $\mathcal{K}^{\uparrow}(\mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{B})) = \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{K}^{\uparrow}(\mathcal{B}))$. If some of the factors that appear in (3.2) are fully pointed, then some of the $\times$-symbols have to be replaced by $\bullet$, but the calculation remains essentially the same. If some of the data do not match, then both $(\mathcal{A} \boxtimes \mathcal{B}) \boxtimes \mathcal{C}$ and $\mathcal{A} \boxtimes (\mathcal{B} \boxtimes \mathcal{C})$ equal $\widehat{\Omega}$. Hence, we have proved associativity for products of pre-generators. The general statement can easily be derived from this special case. This proves associativity.

To prove the $\boxplus$-linearity, suppose we are given $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{C}$ which are all products of pre-generators, i.e.

$$
\begin{aligned}
\mathcal{P} &= \mathcal{P}_1 \times \cdots \times \mathcal{P}_k \\
\mathcal{R} &= \mathcal{R}_1 \times \cdots \times \mathcal{R}_l \\
\mathcal{C} &= \mathcal{C}_1 \times \cdots \times \mathcal{C}_m.
\end{aligned}
$$

The product $\mathcal{P} \boxtimes (\mathcal{R} \boxplus \mathcal{C})$ equals $\widehat{\Omega}$ if the data of $\mathcal{R}$ and $\mathcal{C}$ do not match. This means there is a vertex s such that one of the following equalities is violated

$$
\mathfrak{m}^{\mathcal{C}}(s) = \mathfrak{m}^{\mathcal{R}}(s) \tag{3.3}
$$

$$
\mathfrak{m}_{\mathcal{C}}(s) = \mathfrak{m}_{\mathcal{R}}(s). \tag{3.4}
$$

We would like to see that in each of these cases we have

$$
(\mathcal{P} \boxtimes \mathcal{R}) \boxplus (\mathcal{P} \boxtimes \mathcal{C}) = \widehat{\Omega}. \tag{3.5}
$$

Assuming a violation of (3.3), we may suppose without loss of generality the inequality $\mathfrak{m}^{\mathcal{R}}(s) > \mathfrak{m}^{\mathcal{C}}(s)$ to hold. Recall from its definition that

$$
\begin{aligned}
\mathcal{P} \boxtimes \mathcal{R} &= \begin{cases} (\mathcal{P} \boxtimes \mathcal{R}_1) \bullet \mathcal{R}_2 \times \cdots \times \mathcal{R}_l & \text{, if } \mathcal{K}^{\uparrow}(\mathcal{P}), \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{R}_1) \in \mathbb{F} \\ (\mathcal{P} \boxtimes \mathcal{R}_1) \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_l & \text{otherwise} \end{cases} \\
\mathcal{P} \boxtimes \mathcal{C} &= \begin{cases} (\mathcal{P} \boxtimes \mathcal{C}_1) \bullet \mathcal{C}_2 \times \cdots \times \mathcal{C}_m & \text{, if } \mathcal{K}^{\uparrow}(\mathcal{P}), \mathcal{K}^{\downarrow, F^{\uparrow}}(\mathcal{C}_1) \in \mathbb{F} \\ (\mathcal{P} \boxtimes \mathcal{C}_1) \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_m & \text{otherwise} \end{cases}.
\end{aligned}
$$

Supposing that the data of $\mathcal{P}$ and $\mathcal{R}_1$ as well as the data of $\mathcal{P}$ and $\mathcal{C}_1$ match (the other cases are uninteresting), this implies

$$
\mathfrak{m}^{\mathcal{P} \boxtimes \mathcal{R}_1}(s) - \mathfrak{m}^{\mathcal{R}_1}(s) = \mathfrak{m}^{\mathcal{P} \boxtimes \mathcal{C}_1}(s) - \mathfrak{m}^{\mathcal{C}_1}(s).
$$

By Lemma 3.4 we conclude $\mathfrak{m}^{\mathcal{P}\boxtimes\mathcal{R}}(s) > \mathfrak{m}^{\mathcal{P}\boxtimes\mathcal{C}}(s)$ which shows that (3.5) holds in this case. Assuming a violation of (3.4), there are two cases to consider. If either the data of $\mathcal{P}$ and $\mathcal{R}_1$, or the data of $\mathcal{P}$ and $\mathcal{C}_1$ do not match, the equality (3.5) is satisfied. If the data of $\mathcal{P}$ and $\mathcal{R}_1$ as well as the data of $\mathcal{P}$ of $\mathcal{C}_1$ match, we have

$$\mathfrak{m}_{\mathcal{P}\boxtimes\mathcal{R}}(s) = \mathfrak{m}_{\mathcal{R}}(s) > \mathfrak{m}_{\mathcal{C}}(s) = \mathfrak{m}_{\mathcal{P}\boxtimes\mathcal{C}}(s)$$

which implies (3.5).

Supposing that both (3.3) and (3.4) are fulfilled, we have the following chain of equalities

$$\mathcal{P} \boxtimes (\mathcal{R} \boxplus \mathcal{C}) = \mathcal{P} \boxtimes (\mathcal{R} + \mathcal{C}) = (\mathcal{P} \boxtimes \mathcal{R}) \boxplus (\mathcal{P} \boxtimes \mathcal{C}).$$

This shows that the product $\boxtimes$ is $\boxplus$-linear in the second component. An analogue line of arguments shows the linearity in the first component. $\qquad\square$

Definition 3.7. Let A be a R-semialgebra and B a T-semialgebra. A map $\phi\colon A \longrightarrow B$ is called a (R,T)-**semialgebra morphism** if there is a morphism $\psi\colon R \longrightarrow T$ such that

$$\begin{aligned}
\phi(r \bullet a) &= \psi(r) \bullet \phi(a) \\
\phi(a + b) &= \phi(a) + \phi(b) \\
\phi(a \times b) &= \phi(a) \times \phi(b)
\end{aligned}$$

for $a \in A$, $b \in B$ and $r \in R$. If $R = T$, then we call ϕ an R-**semialgebra morphism** and in case the coefficients are defined in the context just **semialgebra morphism**.

Define a map

(3.6) $$\Phi\colon (\widehat{\mathcal{T}}, +, \times) \longrightarrow (\widehat{\mathcal{T}}, \boxplus, \boxtimes)$$

by sending $+$ to $\boxplus$, $\times$ to $\boxtimes$ and by requiring that $\Phi(\widehat{\Omega}) = \widehat{\Omega}$. This is a $\mathbb{F}$-semialgebra morphism. The image of this morphism, which is a $\mathbb{F}$-semialgebra, will be the set we are interested in.

Definition 3.8. We denote by $\mathcal{G}$ the set of all generators and denote by $\mathcal{Q}$ the subalgebra of $\mathcal{T}$ generated by the elements of $\mathcal{G}$. We denote by $\mathcal{S}$ the set $\Phi(\mathcal{Q}) \cap \mathcal{T}$ and by $\widehat{\mathcal{S}}$ the set $\Phi(\mathcal{Q})$. We call $\widehat{\mathcal{S}}$ the **symbol algebra**.

Alternatively, we can think of $\widehat{\mathcal{S}}$ as the $\mathbb{F}$-subsemialgebra of $(\widehat{\mathcal{T}}, \boxplus, \boxtimes)$ which is generated by the elements of $\mathcal{G}$.

3.2. **The Differential on the Symbol Algebra.** Recall, that a generator is a moduli space with auxiliary data attached to it. As such, we can look at its codimension-1-boundary. The moduli spaces of Whitney polygons are Gromov compact manifolds, because of the existence of an energy bound which was shown in [4]. Hence, approaching the boundary of a moduli space of Whitney polygons – a priori – disks break and spheres bubble off. In our case, bubbling can be ruled out easily due to the fact that holomorphic spheres in the symmetric product all have non-zero intersection number n_z. Our goal will be to use Gromov's notion of codimension-1-boundary to introduce a map on the symbol algebra,
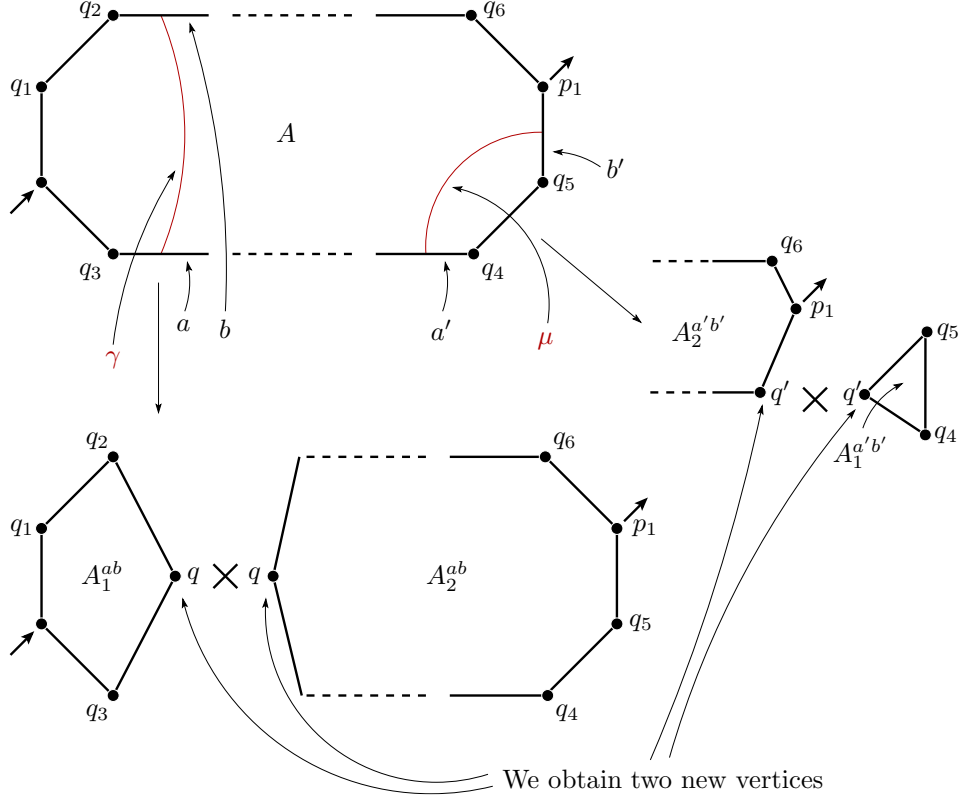
FIGURE 1. Here we see the two cases that might occur in the boundary. The shrinking of γ gives a product of moduli spaces which is a $\boxtimes$-product of generators, whereas the shrinking of μ gives a product of moduli spaces of whom one is a fully pointed pre-generator.

whose square vanishes identically: We define $\partial_{\mathfrak{sh}}(\widehat{\Omega}) = 0$ for a start. Now suppose we are given a generator $\mathcal{A} = A_{(P^\bullet, F^\downarrow, F^\uparrow)}$. We define $\partial^{\mathrm{codim}\ 1}(\pi(\mathcal{A}))$ as the codimension-1 boundary of $\pi(\mathcal{A})$ (see §3.1 for a definition of π). We know from Gromov compactness that the boundary components are Cartesian products of moduli spaces of Whitney polygons. Each of the boundary components is obtained in the following way. Each pair (a, b) with $a, b \in B'$ such that $a \neq b$ specifies a pair of edges in the polygon. Here, traversing along the boundary in the preferred direction (i.e. given by the orientation induced on the boundary) starting at the flow-out vertex, we first move over b and then over a. Given such a pair (a, b), we choose a path γ from the a-boundary to the b-boundary and then shrink it to a point. The polygon will split into two polygons A_1 and A_2, where each of these polygons carryies a new vertex, v_1 say for A_1 and v_2 say for A_2 (cf. Figure 1). These polygons represent maps ϕ_i, $i = 1, 2$. Furthermore, we know that $\phi_1(v_1) = \phi_2(v_2)$ is a point in $\mathbb{T}_a \cap \mathbb{T}_b$. We denote this point by $\boldsymbol{q}$. The associated boundary component of $\pi(A)$ (see §3.1 for a definition of π) is the Cartesian product of two moduli spaces we denote by $A_1^{ab;\boldsymbol{q}}$ and $A_2^{ab;\boldsymbol{q}}$. Hence, we

have that

$$\partial^{\mathrm{codim}\,1}\pi(\mathcal{A}) = \bigsqcup_{\substack{a,b\in B' \\ a\neq b}} \bigsqcup_{q\in\mathbb{T}_a\cap\mathbb{T}_b} A_1^{ab;\boldsymbol{q}} \times A_2^{ab;\boldsymbol{q}}.$$

Let V_i be the set of vertices of $A_i^{ab;\boldsymbol{q}}$. We decorate the vertices in $V_i\setminus\{v_i\}$ with the data from the corresponding vertices of $\mathcal{A}$. In this way, we define $P_i^\bullet$, $F_i^\downarrow$ and $F_i^\uparrow$ which provide decorations for all vertices but v_i. So, it remains to give an algorithm which attaches a decoration to the vertex v_i: We know that $\#(\mathfrak{D}(F^\uparrow(\mathcal{A})))=1$. Thus, either $A_1^{ab;\boldsymbol{q}}$ or $A_2^{ab;\boldsymbol{q}}$ has a flow-out vertex. Without loss of generality, we assume it is $A_2^{ab;\boldsymbol{q}}$. There are two cases we have to consider which are illustrated in Figure 1.

Case 1. In the first case $A_1^{ab;\boldsymbol{q}}$ has a vertex which is a flow-in vertex. An example is illustrated in the lower part of Figure 1. We obtain the pictured splitting by shrinking the curve γ to a point. We *attach* to v_1 a flow-out arrow and to v_2 a flow-in arrow: More precisely, define

(3.7) $$\{\{(a,b),\boldsymbol{q}\}\}\cup F_1^\uparrow$$

and – by abuse of notation – denote this set by $F_1^\uparrow$. Furthermore, we define

(3.8) $$\{(b,a)\}\cup F_2^\downarrow$$

and denote this set by $F_2^\downarrow$. We remove all points which were attached to the vertices of $A_i^{ab;\boldsymbol{q}}$ and then decorate it with the data $(P_i^\bullet, F_i^\downarrow, F_i^\uparrow)$. This provides us with decorated moduli spaces we denote by $\mathcal{A}_i^{ab;\boldsymbol{q}}$, $i=1,2$. Observe that with this construction both $\mathcal{A}_1^{ab;\boldsymbol{q}}$ and $\mathcal{A}_2^{ab;\boldsymbol{q}}$ are generators.

Case 2. In the second case, $\mathcal{A}_1^{ab;\boldsymbol{q}}$ carries no flow-in vertex. An illustration is given in the right part of Figure 1. Here, we contracted the curve μ. As in the first case, we define sets of data $(P_i^\bullet, F_i^\downarrow, F_i^\uparrow)$ which are given by extracting the data from $\mathcal{A}$ that correspond to the old vertices. Instead of the modifications given in (3.7) and (3.8) we perform the following modifications: Set

(3.9) $$\{\{(a,b),\boldsymbol{q}\}\}\cup P_1^\bullet \quad \text{and} \quad \{\{(b,a),\boldsymbol{q}\}\}\cup P_2^\bullet$$

and denote the corresponding set with $P_i^\bullet$, $i=1,2$. We then continue as in the first case: We remove the points attached to the vertices of $A_i^{ab;\boldsymbol{q}}$, $i=1,2$, and then decorate it with the data $(P_i^\bullet, F_i^\downarrow, F_i^\uparrow)$ to obtain pre-generators $\mathcal{A}_i^{ab;\boldsymbol{q}}$, $i=1,2$. Observe that in this case $\mathcal{A}_2^{ab;\boldsymbol{q}}$ is a generator but $\mathcal{A}_1^{ab;\boldsymbol{q}}$ is not. It is a fully pointed pre-generator and, therefore, an element of the coefficient algebra $\mathbb{F}$. The Cartesian product $A_1^{ab;\boldsymbol{q}} \times A_2^{ab;\boldsymbol{q}}$ thus, should be thought of as realized by $\mathcal{A}_1^{ab;\boldsymbol{q}} \bullet \mathcal{A}_2^{ab;\boldsymbol{q}}$.

We apply this algorithm to all components of $\partial^{\mathrm{codim}\,1}(\pi(\mathcal{A}))$. A pair of edges is called **nice** if for the associated boundary component the first case of the algorithm applies. For

$\mathcal{A} \in \mathcal{G}$ we define

$$\partial_{\mathfrak{sh}}(\mathcal{A}) = \Big(\boxplus_{\substack{(a,b) \text{ nice} \\ \boldsymbol{q} \in \mathbb{T}_a \cap \mathbb{T}_b}} \mathcal{A}_1^{ab;\boldsymbol{q}} \boxtimes \mathcal{A}_2^{ab;\boldsymbol{q}} \Big) \boxplus \Big(\boxplus_{\substack{(a,b) \text{ not nice} \\ \boldsymbol{q} \in \mathbb{T}_a \cap \mathbb{T}_b}} \mathcal{A}_1^{ab;\boldsymbol{q}} \bullet \mathcal{A}_2^{ab;\boldsymbol{q}} \Big).$$

Observe that if $\mathcal{A}$ is a fully pointed pre-generator, the above arguments carry over verbatim to define an assignment $\partial_{\mathbb{F}}(\mathcal{A})$ by

$$\partial_{\mathbb{F}}(\mathcal{A}) = \sum_{\substack{(a,b) \text{ not nice} \\ \boldsymbol{q} \in \mathbb{T}_a \cap \mathbb{T}_b}} \mathcal{A}_1^{ab;\boldsymbol{q}} \bullet \mathcal{A}_2^{ab;\boldsymbol{q}}.$$

So, we obtain a map $\partial_{\mathbb{F}} \colon \mathbb{F} \longrightarrow \mathbb{F}$, by requiring that $\partial_{\mathbb{F}}$ is $\boxplus$-linear and fulfills a Leibniz-rule, i.e. for elements $\mathcal{A}, \mathcal{B} \in \mathbb{F}$ we have

$$\partial_{\mathbb{F}}(\mathcal{A} \bullet \mathcal{B}) = \partial_{\mathbb{F}}(\mathcal{A}) \bullet \mathcal{B} + \mathcal{A} \bullet \partial_{\mathbb{F}}(\mathcal{B}).$$

We extend the definition of $\partial_{\mathfrak{sh}}$ from the generators to the symbol algebra by requiring that for given $\boxtimes$-products $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of generators and elements $f_1, \ldots, f_k \in \mathbb{F}$ the following equalities hold:

$$(3.10) \qquad \begin{aligned} \partial_{\mathfrak{sh}}(f_1 \bullet \mathcal{A}_1) &= \partial_{\mathbb{F}}(f_1) \bullet \mathcal{A}_1 \boxplus f_1 \bullet \partial_{\mathfrak{sh}}(\mathcal{A}_1) \\ \partial_{\mathfrak{sh}}\Big(\boxplus_{i=1,\ldots,k} f_i \bullet \mathcal{A}_i \Big) &= \boxplus_{i=1,\ldots,k} \partial_{\mathfrak{sh}}(f_i \bullet \mathcal{A}_i) \\ \partial_{\mathfrak{sh}}(\mathcal{A}_1 \boxtimes \mathcal{A}_2) &= \partial_{\mathfrak{sh}}(\mathcal{A}_1) \boxtimes \mathcal{A}_2 \boxplus \mathcal{A}_1 \boxtimes \partial_{\mathfrak{sh}}(\mathcal{A}_2). \end{aligned}$$

This uniquely extends the map onto the symbol algebra.

Theorem 3.9. *The map $\partial_{\mathfrak{sh}} \colon \widehat{\mathcal{S}} \longrightarrow \widehat{\mathcal{S}}$ is a differential, i.e. $\partial_{\mathfrak{sh}} \circ \partial_{\mathfrak{sh}} = 0$, and $\partial_{\mathfrak{sh}}(\mathcal{S}) \subset \mathcal{S}$.*

Proof. To see that it is a differential, first observe that this statement is nontrivial for those generators only whose underlying moduli space is 1-dimensional. In this case, we have that

$$\begin{aligned} \partial_{\mathfrak{sh}}^2(\mathcal{A}) &= \partial_{\mathfrak{sh}}\Big(\boxplus_{\substack{a,b \in B' \text{ nice} \\ q \in \mathbb{T}_a \cap \mathbb{T}_b}} \mathcal{A}_1^{ab;\boldsymbol{q}} \boxtimes \mathcal{A}_2^{ab;\boldsymbol{q}} \boxplus \boxplus_{\substack{a',b' \in B' \text{ not nice} \\ q \in \mathbb{T}_{a'} \cap \mathbb{T}_{b'}}} \mathcal{B}_1^{a'b';\boldsymbol{q}} \bullet \mathcal{B}_2^{a'b';\boldsymbol{q}} \Big) \\ &= \boxplus_{\substack{a,b \in B' \text{ nice} \\ q \in \mathbb{T}_a \cap \mathbb{T}_b}} \Big(\partial_{\mathfrak{sh}}(\mathcal{A}_1^{ab;\boldsymbol{q}}) \boxtimes \mathcal{A}_2^{ab;\boldsymbol{q}} \boxplus \mathcal{A}_1^{ab;\boldsymbol{q}} \boxtimes \partial_{\mathfrak{sh}}(\mathcal{A}_2^{ab;\boldsymbol{q}}) \Big) \\ &\quad \boxplus \boxplus_{\substack{a',b' \in B' \text{ not nice} \\ q \in \mathbb{T}_{a'} \cap \mathbb{T}_{b'}}} \Big(\partial_{\mathbb{F}}(\mathcal{B}_1^{a'b';\boldsymbol{q}}) \bullet \mathcal{B}_2^{a'b';\boldsymbol{q}} \boxplus \mathcal{B}_1^{a'b';\boldsymbol{q}} \bullet \partial_{\mathfrak{sh}}(\mathcal{B}_2^{a'b';\boldsymbol{q}}) \Big) \\ &= 0 \end{aligned}$$

where the third equality holds since $\mathcal{A}_1^{ab;\boldsymbol{q}}$, $\mathcal{A}_2^{ab;\boldsymbol{q}}$, $\mathcal{B}_1^{a'b';\boldsymbol{q}}$ and $\mathcal{B}_2^{a'b';\boldsymbol{q}}$ are 0-dimensional and, thus, $\partial_{\mathfrak{sh}}(\mathcal{A}_1^{ab;\boldsymbol{q}})$, $\partial_{\mathfrak{sh}}(\mathcal{A}_2^{ab;\boldsymbol{q}})$, $\partial_{\mathbb{F}}(\mathcal{B}_1^{a'b';\boldsymbol{q}})$ and $\partial_{\mathfrak{sh}}(\mathcal{B}_2^{a'b';\boldsymbol{q}})$ vanish. Given generators $\mathcal{B}_1, \ldots, \mathcal{B}_k$,

we would like to check that

$$\partial_{\mathfrak{sh}}^2\Big(\boxtimes_{i=1,\ldots,k} \mathcal{B}_i\Big) = 0.$$

This can be derived by induction on k by applying the fact that $\partial_{\mathfrak{sh}}^2(\mathcal{B}_i) = 0$. Now let $\mathcal{B}$ be a $\boxtimes$-product of generators and $f \in \mathbb{F}$, then we have the following chain of equalities.

$$
\begin{aligned}
\partial_{\mathfrak{sh}}^2(f \bullet \mathcal{B}) &= \partial_{\mathfrak{sh}}(\partial_{\mathbb{F}}(f) \bullet \mathcal{B}) \boxplus \partial_{\mathfrak{sh}}(f \bullet \partial_{\mathfrak{sh}}(\mathcal{B})) \\
&= (\partial_{\mathbb{F}})^2(f) \bullet \mathcal{B} \boxplus 2\partial_{\mathbb{F}}(f) \bullet \partial_{\mathfrak{sh}}(\mathcal{B}) \boxplus f \bullet (\partial_{\mathbb{F}})^2(\mathcal{B}) \\
&= (\partial_{\mathbb{F}})^2(f) \bullet \mathcal{B} \\
&= \Big(\sum_{(a,b),\boldsymbol{q}} \partial_{\mathbb{F}}(\mathcal{B}_1^{ab;\boldsymbol{q}} \bullet \mathcal{B}_2^{ab;\boldsymbol{q}})\Big) \bullet \mathcal{B} \\
&= \Big(\sum_{(a,b),\boldsymbol{q}} (\partial_{\mathbb{F}}(\mathcal{B}_1^{ab;\boldsymbol{q}}) \bullet \mathcal{B}_2^{ab;\boldsymbol{q}} + \mathcal{B}_1^{ab;\boldsymbol{q}} \bullet \partial_{\mathbb{F}}(\mathcal{B}_2^{ab;\boldsymbol{q}}))\Big) \bullet \mathcal{B} \\
&= 0.
\end{aligned}
$$

The first and the second equality follow from the equations given in (3.10). The third equality is derived using both the commutative $\mathbb{Z}_2$-algebra structure of $\mathbb{F}$ and the vanishing of $\partial_{\mathfrak{sh}}^2$ on products of generators. The fourth and the fifth equality are immediate from the definition of $\partial_{\mathbb{F}}$ and the sixth equality rests on the fact that the $\mathcal{B}_i^{ab;q}$, for $i = 1, 2$, are 0-dimensional. Combining all results, we see that $\partial_{\mathfrak{sh}}^2 = 0$.

The second statement is a consequence of the definition of both the symbol algebra and the map $\partial_{\mathfrak{sh}}$. □

Now we have the elements ready to define our object of interest.

Definition 3.10. We define the **symbol homology** $\mathfrak{sh}_*$ as the homology theory of the chain complex $(\widehat{\mathcal{S}}, \partial_{\mathfrak{sh}})$.

In fact, as the symbol algebra, the symbol homology carries the structure of a semialgebra.

Proposition 3.11. *The map $\partial_{\mathbb{F}}\colon \mathbb{F} \longrightarrow \mathbb{F}$ is a differential. Denote by $\mathfrak{f}_*$ the homology theory $H_*(\mathbb{F}, \partial_{\mathbb{F}})$, then $\mathfrak{sh}_*$ is a $\mathfrak{f}_*$-semialgebra.*

Proof. The vanishing of $\partial_{\mathbb{F}} \circ \partial_{\mathbb{F}}$ follows from the considerations provided in the proof of Theorem 3.9. Given elements $f \in \mathbb{F}$ with $\partial_{\mathbb{F}}(f) = 0$ and $[\mathcal{A}] \in \mathfrak{sh}_*$, we would like to see that $[f \bullet \mathcal{A}]$ just depends on the homology classes of f and $\mathcal{A}$: Suppose we are given an element $g \in \mathbb{F}$, then

$$
\begin{aligned}
(f + \partial_{\mathbb{F}}(g)) \bullet \mathcal{A} &= f \bullet \mathcal{A} \boxplus \partial_{\mathbb{F}}(g) \bullet \mathcal{A} \\
&= f \bullet \mathcal{A} \boxplus \partial_{\mathbb{F}}(g) \bullet \mathcal{A} \boxplus g \bullet \partial_{\mathfrak{sh}}(\mathcal{A}) = f \bullet \mathcal{A} \boxplus \partial_{\mathfrak{sh}}(g \bullet \mathcal{A})
\end{aligned}
$$

where the second equality holds since $\mathcal{A}$ is closed and where the last equality is given by (3.10). For an element $\mathcal{B}$ we have

$$
\begin{aligned}
f \bullet (\mathcal{A} \boxplus \partial_{\mathfrak{sh}}(\mathcal{B})) &= f \bullet \mathcal{A} \boxplus f \bullet \partial_{\mathfrak{sh}}(\mathcal{B}) \\
&= f \bullet \mathcal{A} \boxplus f \bullet \partial_{\mathfrak{sh}}(\mathcal{B}) \boxplus \partial_{\mathbb{F}}(f) \bullet \mathcal{B} = f \bullet \mathcal{A} \boxplus \partial_{\mathfrak{sh}}(f \bullet \mathcal{B}),
\end{aligned}
$$

where the second equality holds since $\partial_{\mathbb{F}}(f) = 0$ and where the last equality is given by applying (3.10). This shows that the product $\bullet$ descends to a map

$$\bullet \colon \mathfrak{f}_* \times \mathfrak{sh}_* \longrightarrow \mathfrak{sh}_*.$$

To show that $\boxplus$ and $\boxtimes$ descend to maps on the symbol homology we can apply arguments standard in algebraic topology: this proof follows the same lines as the proof which shows that wedging on differential forms induces a product on cohomology. Therefore, we omit these arguments. $\qquad\square$

3.3. **Symbol Homology and Floer Homology.** Suppose we have fixed a set B of attaching circles. For simplicity we will work with $\mathbb{Z}_2$-coefficients. To every element $a = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in B^{\times 2} \backslash \Delta_2$ we can associate the $\mathbb{Z}_2$-vector space $\widehat{\mathrm{CF}}_a = \widehat{\mathrm{CF}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. By building tensor products of these $\widehat{\mathrm{CF}}_a$'s we can construct a wide variety of $\mathbb{Z}_2$-vector spaces. Denote by V and W two of them. We define $\mathbb{MOR}$ as the union of all morphisms between V and W, where V and W vary among all vector spaces we can define as described above. We set

$$\widehat{\mathbb{MOR}} = \mathbb{MOR} \cup \{\widehat{\Omega}\}$$

and equip it with a semialgebra structure as follows: We define a map

$$\widehat{+} \colon \widehat{\mathbb{MOR}} \times \widehat{\mathbb{MOR}} \longrightarrow \widehat{\mathbb{MOR}}$$

by sending a pair $f, g \in \mathbb{MOR}$ to $f + g$ if the source and destination of f and g agree and we send the pair to $\widehat{\Omega}$, otherwise. If the source and destination agree, we say that **the data of f and g match**. In addition, we define $\widehat{\Omega} \widehat{+} f = \widehat{\Omega}$ for every $f \in \widehat{\mathbb{MOR}}$. Furthermore, we define a map

$$\widehat{\circ} \colon \widehat{\mathbb{MOR}} \times \widehat{\mathbb{MOR}} \longrightarrow \widehat{\mathbb{MOR}}$$

in the following way: Suppose we are given a pair $f, g \in \mathbb{MOR}$. If the source of f agrees with the destination of g, then we define $f \widehat{\circ} g = f \circ g$. If f and g are of the form

$$f \colon C_1 \otimes \cdots \otimes C_k \longrightarrow D_i$$
$$g \colon D_1 \otimes \cdots \otimes D_i \otimes \cdots \otimes D_l \longrightarrow E$$

with $C_1, \ldots, C_k, D_1, \ldots, D_l, E$ all Heegaard Floer chain complexes, then we define

$$g \widehat{\circ} f = g \circ (\mathrm{id}_{D_1 \otimes \cdots \otimes D_{i-1}} \otimes f \otimes \mathrm{id}_{D_{i+1} \otimes \cdots \otimes D_l}).$$

If none of the above cases apply, we set $f \widehat{\circ} g = \widehat{\Omega}$. If $f \widehat{\circ} g$ does not equal $\widehat{\Omega}$ we say that **the data of f and g match**. Finally, for every $f \in \widehat{\mathbb{MOR}}$ we define $\widehat{\Omega} \widehat{\circ} f = f \widehat{\circ} \widehat{\Omega} = \widehat{\Omega}$.

Lemma 3.12. *The triple $(\widehat{\mathbb{MOR}}, \widehat{+}, \widehat{\circ})$ is a $\mathbb{Z}_2$-semialgebra.*

The proof rests on the fact that the composition of maps is bilinear with respect to taking sums of maps. Furthermore, taking sums is an associative operation.

Proof. Given three maps f, g, $h \in \mathbb{MOR}$, the triple sum $(f \widehat{+} g) \widehat{+} h$ is not sent to $\widehat{\Omega}$ if any only if all the maps have matching data (in the sense defined above). The same holds for $f \widehat{+} (g \widehat{+} h)$. Thus, we get the equality

$$(f \widehat{+} g) \widehat{+} h = f \widehat{+} (g \widehat{+} h).$$

The commutativity of $\widehat{+}$ follows immediately from its definition and the fact that taking sums of maps is commutative. A similar discussion shows that $\widehat{\circ}$ is associative. For $f, g, h \in \widehat{\mathbb{MOR}}$ the composition $f \widehat{\circ} (g \widehat{+} h)$ equals $\widehat{\Omega}$ unless the destination and source of both g and h match and the destination of g equals the source of f. If all data are matching, then we either have

$$f \widehat{\circ} (g \widehat{+} h) = f \circ (g + h) = f \circ g + f \circ h = f \widehat{\circ} g \widehat{+} f \widehat{\circ} h,$$

or

$$
\begin{aligned}
f \widehat{\circ} (g \widehat{+} h) &= f \circ (\mathrm{id}_A \otimes (g + h) \otimes \mathrm{id}_B) \\
&= f \circ (\mathrm{id}_A \otimes g \otimes \mathrm{id}_B) + f \circ (\mathrm{id}_A \otimes h \otimes \mathrm{id}_B) \\
&= f \widehat{\circ} g \widehat{+} f \widehat{\circ} h
\end{aligned}
$$

for suitable tensor products A and B of Heegaard Floer chain complexes. If the data do not match, then $f \widehat{\circ} (g \widehat{+} h)$ equals $\widehat{\Omega}$. However, in this case the same is true for $f \widehat{\circ} g \widehat{+} f \widehat{\circ} h$. $\square$

Define a map

$$(3.11) \qquad\qquad\qquad\qquad \mathfrak{ct}_{\mathbb{F}} \colon \mathbb{F} \longrightarrow \mathbb{Z}_2$$

by sending fully pointed pre-generators $\mathcal{A}$ to $\mathfrak{ct}_{\mathbb{F}}(\mathcal{A}) = \#(\pi(\mathcal{A}))$ (see §3.1 for a definition of π) and extending $\mathfrak{ct}_{\mathbb{F}}$ as a $\mathbb{Z}_2$-algebra morphism to $\mathbb{F}$. This *counting operation* has a natural counterpart on the symbol algebra which we explain in the following: Given a generator $\mathcal{A} = A_{(P^\bullet, F^\downarrow, F^\uparrow)}$, we introduce the following notation. By $\widehat{\mathrm{CF}}_{F^\downarrow}$ we define the tensor product of Heegaard Floer chain modules determined by the boundary conditions at the vertices in $F^\downarrow$. Similarly, we define $\widehat{\mathrm{CF}}_{F^\uparrow}$. First of all , we define $\mathfrak{ev}(\widehat{\Omega})$ to be the element $\widehat{\Omega} \in \widehat{\mathbb{MOR}}$. Second of all, suppose we are given a generator $\mathcal{A} = A_{(P^\bullet, F^\downarrow, F^\uparrow)}$ with $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k$ the slot points in $P^\bullet$ and $\boldsymbol{r}$ the slot point in $F^\uparrow$, denote by $\underline{\boldsymbol{q}}$ the element $(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k)$. A map

$$\mathfrak{ev}(\mathcal{A}) \colon \widehat{\mathrm{CF}}_{F^\downarrow} \longrightarrow \widehat{\mathrm{CF}}_{F^\uparrow}$$

is defined by sending a generator $\underline{\boldsymbol{x}} = \boldsymbol{x}_1 \otimes \cdots \otimes \boldsymbol{x}_f$ of $\widehat{\mathrm{CF}}_{F^\downarrow}$ to

$$\mathfrak{ev}(\mathcal{A})(\underline{\boldsymbol{x}}) = \# A(\underline{\boldsymbol{x}}, \underline{\boldsymbol{q}}, \boldsymbol{r}) \cdot \boldsymbol{r}$$

and extending as a linear map of $\mathbb{Z}_2$-vector spaces. Now, for $i = 1, \ldots, k$, let $\mathcal{A}_i$ be a $\boxtimes$-product of generators and denote by f_i an element of $\mathbb{F}$. Then, we require the following equalities

$$(3.12) \qquad
\begin{aligned}
\mathfrak{ev}(f_1 \bullet \mathcal{A}_1) &= \mathfrak{ct}_{\mathbb{F}}(f_1) \cdot \mathfrak{ev}(\mathcal{A}_1) \\
\mathfrak{ev}\Big(\boxplus_{i=1,\ldots,k} f_i \bullet \mathcal{A}_i \Big) &= \textstyle\sum_{i=1,\ldots,k} \mathfrak{ct}_{\mathbb{F}}(f_i) \cdot \mathfrak{ev}(\mathcal{A}_i) \\
\mathfrak{ev}(\mathcal{A}_1 \boxtimes \mathcal{A}_2) &= \mathfrak{ev}(\mathcal{A}_2) \widehat{\circ} \mathfrak{ev}(\mathcal{A}_1)
\end{aligned}
$$

These definitions provide a map

$$\mathfrak{ev} \colon (\widehat{\mathcal{S}}, \boxplus, \boxtimes) \longrightarrow (\widehat{\mathbb{MOR}}, \widehat{+}, \widehat{\circ})$$

which is uniquely determined by the above.

Proposition 3.13. *The map $\mathfrak{cv}$ vanishes on boundaries, i.e. $\mathfrak{cv} \circ \partial_{\mathfrak{sh}} = 0$. Hence, it descends to map $\mathfrak{cv}_* \colon \mathfrak{sh}_* \longrightarrow \widehat{\mathbb{MOR}}$ which is a $(\mathfrak{f}_*, \mathbb{Z}_2)$-morphism of semialgebras.*

Proof. The vanishing of $\mathfrak{cv} \circ \partial_{\mathfrak{sh}}$ is a consequence of the fact that 1-dimensional manifolds have an even number of boundary components. Hence, $\mathfrak{cv}$ induces a map $\mathfrak{cv}_*$ on the symbol homology. The induced map $\mathfrak{cv}_*$ is a $(\mathfrak{f}_*, \mathbb{Z}_2)$-morphism of semialgebras since $\mathfrak{cv}$ is a $(\mathbb{F}, \mathbb{Z}_2)$-morphism of semialgebras. $\qquad\square$

Observe that by construction, every map f between Heegaard Floer chain complexes that is defined by counting holomorphic polygons with suitable boundary conditions admits a preferred element $\mathfrak{s}_f$ in the symbol algebra such that $\mathfrak{cv}(\mathfrak{s}_f) = f$. We call $\mathfrak{s}_f$ the **canonical symbol** of f. Sometimes, by abuse of notation, we will also refer to $[\mathfrak{s}] \in \mathfrak{sh}_*$ as the canonical symbol of f.

4. Filtered Symbol Homology

Suppose we are given a set B of attaching circles. In §3 our focus lay on moduli spaces of Whitney polygons ϕ with $n_z(\phi) = 0$. Fixing an additional point w of the Heegaard surface Σ that lies in the complement of the attaching circles given in B, we may look at polygons ϕ as before, with the additional condition $n_w(\phi) = 0$ imposed. We call the associated moduli spaces w-**filtered** to distinguish them from the moduli spaces used in §3. As a path of almost complex structures we choose one which is w-respectful (see §2.1). We use the w-filtered spaces to define the notions of pre-generators, generators and fully pointed pre-generators the same way we did in Definition 3.1 and denote by $\mathcal{G}^w$ the set of generators. Then, following the construction procedure from §3.1 and §3.2, we define the w-**filtered symbol algebra** $\widehat{\mathcal{S}}^w$. The associated homology theory is denoted by $\mathfrak{sh}_*^w$ and called w-**filtered symbol homology**. To fix notation, we introduce the following notational conventions: We denote by $\mathbb{F}^w$ the coefficient algebra and denote by $\mathcal{T}^w$ the non-commutative polynomial $\mathbb{F}^w$-algebra generated by the set of pre-generators which are not fully pointed. And finally, write $\mathfrak{f}_*^w$ for $H_*(\mathbb{F}^w, \partial_{\mathbb{F}^w})$, the coefficient algebra of the w-filtered symbol homology.

4.1. **The Filtering Morphism.** Given a moduli space A of polygons, we define the w-filtered space as
$$A^w = \big\{ \phi \in A \,|\, n_w(\phi) = 0 \big\}.$$
With this in place, we construct a map by assigning to a pre-generator $\mathcal{A} = A_{(P^\bullet, F^\downarrow, F^\uparrow)}$ the element $\mathfrak{F}(\mathcal{A}) = (A^w)_{(P^\bullet, F^\downarrow, F^\uparrow)}$ and extending to a map
$$\mathfrak{F} \colon (\mathcal{T}, +, \times) \longrightarrow (\mathcal{T}^w, +, \times)$$
as a $(\mathbb{F}, \mathbb{F}^w)$-morphism of algebras, where $\mathfrak{F}(\mathcal{C} \bullet \mathcal{A}) = (C^w)_{(P^\bullet, F^\downarrow, F^\uparrow)} \bullet \mathcal{A}$ for a fully pointed pre-generator $\mathcal{C} = C_{(P^\bullet, F^\downarrow, F^\uparrow)}$ and pre-generator $\mathcal{A}$. It is easy to see that $\mathfrak{F}$ sends generators to generators. More precisely, $\mathfrak{F}$ restricts to a bijection between $\mathcal{G}$ and $\mathcal{G}^w$. Hence, with the convention $\mathfrak{F}(\widehat{\Omega}) = \widehat{\Omega}$, the map $\mathfrak{F}$ restricts to a map
$$\mathfrak{F} \colon \widehat{\mathcal{S}} \longrightarrow \widehat{\mathcal{S}}^w$$

between the symbol algebras.

Theorem 4.1. *The map $\mathfrak{F}\colon \widehat{\mathcal{S}} \longrightarrow \widehat{\mathcal{S}}^w$ is a $(\mathbb{F}, \mathbb{F}^w)$-morphism of semialgebras. Furthermore, if $\mathcal{J}_s$ is w-respectful, then the map $\mathfrak{F}$ is a chain map and, thus, induces a map*

$$\mathfrak{F}_*\colon \mathfrak{sh}_* \longrightarrow \mathfrak{sh}_*^w.$$

which is a $(\mathfrak{f}_*, \mathfrak{f}_*^w)$-morphism of semialgebras.

We both call $\mathfrak{F}$ and the induced map $\mathfrak{F}_*$ the **filtering morphism**. This morphism is the main object of interest in this section.

Proof. Since $\mathcal{J}_s$ respects the point w, we know that $V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$ is a complex submanifold of $\mathrm{Sym}^g(\Sigma)$. Thus, every intersection of a $\mathcal{J}_s$-holomorphic polygon with V_w is positive. Furthermore, we know that the intersection number n_w is a homotopical invariant and behaves additive under splicing. Thus, the w-filtered boundary of a moduli space A equals the boundary of the w-filtered moduli space A^w, i.e.

$$\left(\partial^{\mathrm{codim}\,1}(A)\right)^w = \partial^{\mathrm{codim}\,1}(A^w).$$

Interpreted in the language of the symbol algebra, this translates into $\mathfrak{F} \circ \partial_{\mathfrak{sh}} = \partial_{\mathfrak{sh}}^w \circ \mathfrak{F}$. Thus, $\mathfrak{F}$ is chain. To see that $\mathfrak{F}$ is a morphism of semialgebras, recall that there is a semialgebra morphism

$$\Phi\colon (\widehat{\mathcal{T}}, +, \times) \longrightarrow (\widehat{\mathcal{T}}, \boxplus, \boxtimes)$$

(see (3.6)) with the following property: The image of $\Phi|_{\mathcal{Q}}$, i.e. the image of Φ restricted to the subalgebra $\mathcal{Q}$ (see Definition 3.8), is the symbol algebra. It is easy to see from its definition that $\Phi|_{\mathcal{Q}\backslash\Phi^{-1}(\widehat{\Omega})}$ is injective. Hence, the following square is commutative.

$$
\begin{array}{ccc}
\mathcal{Q} & \xrightarrow{\;\;\mathfrak{F}\;\;} & \mathcal{Q}^w \\
{\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\
\mathcal{S}(\mathcal{G}) = \Phi(\mathcal{Q})\backslash\widehat{\Omega} & \xrightarrow{\;\;\mathfrak{F}\;\;} & \Phi(\mathcal{Q}^w)\backslash\widehat{\Omega} = \mathcal{S}(\mathcal{G}^w)
\end{array}
$$

The map Φ is a morphism of semialgebras and it surjects onto the symbol algebra. Hence, the map $\mathfrak{F}$ restricted to the symbol algebra is a morphism of semialgebras. Since $\mathfrak{F}$ is a $(\mathbb{F}, \mathbb{F}^w)$-morphism of semialgebras, its induced map in homology is a $(\mathfrak{f}_*, \mathfrak{f}_*^w)$-morphism of semialgebras. $\qquad\square$

4.2. **Property P of Morphisms/Symbols.** Denote by $\mathfrak{sh}_*(\mathcal{G})\{X\}$ the non-commutative polynomial algebra in one variable, defined using the sum $\boxplus$ and the product $\boxtimes$. Define $\mathbb{P}(\mathcal{G})$ as the polynomials of degree at least one.

Definition 4.2. For an element $\mathfrak{s} \in \mathfrak{sh}_*(\mathcal{G})$ we say that $\mathfrak{s}$ **has property P** if P is a polynomial in $\mathbb{P}(\mathcal{G})$ with root $\mathfrak{s}$. Furthermore, we say that $\mathfrak{ev}(\mathfrak{s})$ has property P, if $\mathfrak{s}$ has property P.

Observe that the filtering morphism $\mathfrak{F}_*$ extends to a morphism

$$\mathfrak{sh}_*(\mathcal{G})\{X\} \longrightarrow \mathfrak{sh}_*(\mathcal{G}^w)\{X\}, \ p \longmapsto p_{\mathfrak{F}}$$

by defining $\mathfrak{F}_*(X) = X$. As such, this map restricts to a morphism $\mathbb{P}(\mathcal{G}) \longrightarrow \mathbb{P}(\mathcal{G}^w)$.

Theorem 4.3. *Suppose we are given a symbol $\mathfrak{s} \in \mathfrak{sh}_*$. If $\mathfrak{s}$ has property P, then $\mathfrak{F}_*(\mathfrak{s}) \in \mathfrak{sh}_*^w$ has property $P_{\mathfrak{F}}$.*

Proof. This is an immediate application of the fact that $\mathfrak{F}_*$ is a morphism of semialgebras. We have that $P_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s})) = \mathfrak{F}_*(P(\mathfrak{s})) = \mathfrak{F}_*(0) = 0$. $\qquad\square$

A map $f\colon \bigotimes_i \widehat{\mathrm{CF}}_i \longrightarrow \widehat{\mathrm{CF}}'$ between suitable Heegaard Floer chain modules which is defined by counting elements of moduli spaces of Whitney polygons corresponds to a homology class $\mathfrak{s}_f \in \mathfrak{sh}_*$ via the morphism $\mathfrak{ev}_*$, i.e. $\mathfrak{ev}_*(\mathfrak{s}_f) = f$. A property of f can be encoded into a polynomial expression P with coefficients in $\mathbb{MOR}$ such that $P(f) = 0$ if and only if f fulfills the property. Given such a polynomial P is there a method to relate P to a polynomial P' such that $P'(f^w) = 0$? Morphisms between symbol homologies give us a method to do that as Theorem 4.3 indicates (cf. §1 and cf. §5).

5. Examples and Ideas

In this section we communicate some of the ideas behind the construction we gave in the previous sections. The ideas behind the theory are very simple and all the operations we provide are based upon simple algorithms. Although the ideas are simple, writing these concepts down formally turns out to be difficult and extensive. We think that reading this section will help the reader to familiarize with the techniques. In fact, we introduce some notation for indicating the decorations of a moduli space which makes the whole construction intuitive. In the following sections we will use the notation introduced here. Furthermore, we present two explicit calculations of symbol homologies in easy situations (see Example 5.1 and Example 5.2) and two examples which should help indicating in what way the symbol homologies can be of benefit (see Example 5.3 and Example 5.4): One of the benefits of this theory is that it unifies the Floer chain level and the moduli space level into one object (see the discussion below, cf. §6 and §3.3). A consequence of this unification is that it provides a systematic and immediate way to transfer properties between different Floer theoretic settings without difficulty. Proofs of properties which need the moduli space machinery now do not need to be repeated in different settings but can now just be accepted by pointing to our results. In Example 5.4 at the end of this section we give an easy demonstration how this transfer is done when explicitly worked out. This technique will be applied in an invariance proof of cobordism maps between knot Floer homologies (see Theorem 1.2) and for a surgery exact triangle in knot Floer homologies (see Theorem 10.2).

Suppose we are given a set $B = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ of attaching circles in a surface Σ. In the previous sections we decorate moduli spaces with auxiliary data. This is done by the following rules which we exemplify on the moduli space $\mathcal{M}^0_{(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})}$ of holomorphic Whitney

triangles ϕ with Maslov index 0, $n_z(\phi) = 0$ and boundary conditions given by $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$. The boundary conditions specify conditions on the edges of the triangle ϕ. Additionally, we may impose conditions on the vertices of the triangle, i.e. specify points they have to be mapped to. In the literature this is indicated as follows $\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q})$ where $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\beta}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$ and $\boldsymbol{q} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$. Instead of attaching just points to the vertices there are now three different types of decorations. We either attach $\overset{\downarrow}{\cdot}$, $\overset{\uparrow}{\boldsymbol{x}}$ or $\dot{\boldsymbol{x}}$ (see Definition 3.1). The first makes the corresponding vertex a flow-in vertex, the second makes it a flow-out vertex and the third a pointed vertex (see Definition 3.1). The point $\boldsymbol{x}$ is called the slot-point of the vertex. The additional information given by the decorations allow us to interpret moduli spaces in various ways: For instance, the space $\mathcal{A} = \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{y}}, \overset{\uparrow}{\boldsymbol{q}})$ can be interpreted as a map

$$\mathfrak{ev}(\mathcal{A}) \colon \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \longrightarrow \widehat{\mathrm{CF}}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$$

such that $\mathfrak{ev}(\mathcal{A})(\boldsymbol{x}) = \#(\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q})) \cdot \boldsymbol{q}$ (see Proposition 3.13). Put in words, the flow-in vertices determine the source of the map, the flow-out vertices the destination and pointings serve as boundary conditions. In this way, all maps in Heegaard Floer theory (which are defined by counting elements of moduli spaces of Whitney polygons) can be represented as decorated moduli spaces if we find a suitable way to express sums of maps in terms of moduli spaces and if we find a suitable way to express compositions of maps in terms of moduli spaces.

To this end, suppose we are given an additional decorated space $\mathcal{B} = \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{r}}, \overset{\uparrow}{\boldsymbol{q}})$, the map $\mathfrak{ev}(\mathcal{A}) + \mathfrak{ev}(\mathcal{B})$ sends an element $\boldsymbol{x}$ to

$$
\begin{aligned}
(\mathfrak{ev}(\mathcal{A}) + \mathfrak{ev}(\mathcal{B}))(\boldsymbol{x}) &= \mathfrak{ev}(\mathcal{A})(\boldsymbol{x}) + \mathfrak{ev}(\mathcal{B})(\boldsymbol{x}) \\
&= (\#\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) + \#\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{r},\boldsymbol{q})) \cdot \boldsymbol{q} \\
&= \#(\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) \sqcup \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{r},\boldsymbol{q})) \cdot \boldsymbol{q}
\end{aligned}
$$

So, it makes sense to define $\mathcal{A} \boxplus \mathcal{B}$ as the disjoint union of the spaces (see §3). However, there are some difficulties that arise as sources and destinations that are specified by the flow-in vertices of decorated moduli spaces might not be matching. This produces some issues which require some consideration.

In a similar vein, we proceed to get a candidate for a product. Suppose we are given the decorated space $\mathcal{C} = \widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{x}})$. The composition $\mathfrak{ev}(\mathcal{C}) \circ \mathfrak{ev}(\mathcal{A})$ sends an element $\boldsymbol{z}$ to

$$\mathfrak{ev}(\mathcal{C}) \circ \mathfrak{ev}(\mathcal{A})(\boldsymbol{z}) = \#(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{z},\boldsymbol{x})) \cdot \#(\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q}) \cdot \boldsymbol{q}.$$

Since

$$\#(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{z},\boldsymbol{x})) \cdot \#(\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q})) = \#(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{z},\boldsymbol{x}) \times \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{q})),$$

the Cartesian product is the right candidate for the product $\mathcal{C} \boxtimes \mathcal{A}$ (see §3.1). However, the product has to change decorations suitably (cf. Definition 3.3). We define

$$
\begin{aligned}
\mathcal{C} \boxtimes \mathcal{A} &= \widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{x}}) \boxtimes \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{y}}, \overset{\uparrow}{\boldsymbol{q}}) \\
&= \widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{x}}) \times \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}, \overset{\uparrow}{\boldsymbol{q}}).
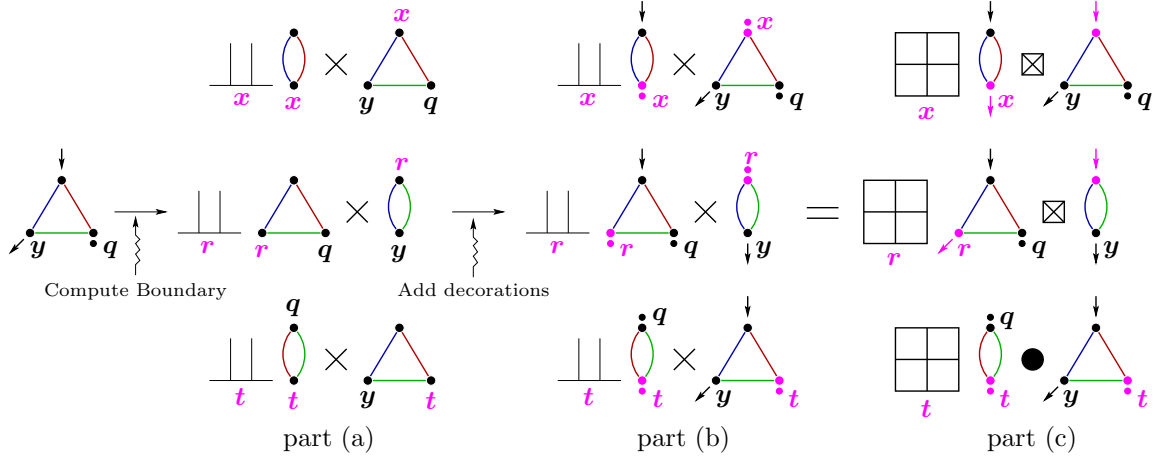\end{aligned}
$$

FIGURE 2. How to define the boundary $\partial_{\mathfrak{sh}}$: We first compute the boundary and then decorate the vertices with data. The black decorations are inherited from the old space and the pink decorations are attached due to the principle that *we need a unique and well-defined flowing direction.*

Put in words, the algorithm goes as follows: Observe that the flow-out vertex of the decorated space $\mathcal{C}$ appears as a flow-in vertex of $\mathcal{A}$ (in the sense of Definition 3.3 (d)). We transform the flow-out vertex of $\mathcal{C}$ to a pointed vertex (by changing its decoration) while keeping the slot-point – $\boldsymbol{x}$ in our case – unchanged. Furthermore, we transform the corresponding flow-in vertex of $\mathcal{A}$ to a pointed vertex and add $\boldsymbol{x}$ as its slot-point (see §3.1). In this way, the product uniquely specifies the source and destination in such a way that it coincides with the source and destination of $\mathfrak{ev}(\mathcal{A}) \circ \mathfrak{ev}(\mathcal{C})$. Again, there are technical difficulties arising in this construction which require some consideration.

To be able to see properties of the Floer homologies in our setting, we have to realize boundaries of moduli spaces as suitable $\boxplus$-sums of $\boxtimes$-products of elements in the symbol algebra (see §3.2). We tried to indicate this in Figure 2: Let us consider the decorated moduli space $\mathcal{M}^1_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{q}}, \overset{\uparrow}{\boldsymbol{y}})$. We first compute the codimension-1 boundary of the space $\mathcal{M}^1_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\cdot \boldsymbol{q}, \boldsymbol{y})$. This is indicated in part (a) of Figure 2. In part (b) of Figure 2 we decorate the spaces with data. Observe that each moduli space in the boundary admits two types of vertices. The (old) vertices which coincide with vertices of the space $\mathcal{M}^1_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\cdot \boldsymbol{q}, \boldsymbol{y})$ and vertices which are new, i.e. generated in the boundary. The old vertices are indicated as black dots in Figure 2 and the new vertices are colored pink. We decorate the black vertices with the same data as the corresponding vertices in $\mathcal{M}^1_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{q}}, \overset{\uparrow}{\boldsymbol{y}})$ and the new vertices according to the principle of having a *unique and well-defined flowing direction.* We point the reader to §3.2 for the description of the algorithm. In this process, the following phenomenon appears which complicates the construction slightly: The bottom of part (b) shows a Cartesian product

$$\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{t}}) \times \mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot}, \dot{\boldsymbol{t}}, \overset{\uparrow}{\boldsymbol{z}})$$

The space $\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}},\dot{\boldsymbol{t}})$ cannot be interpreted as a map using the algorithm presented above for $\mathfrak{ev}$. So, the given Cartesian product cannot be a $\boxtimes$-product since the latter should correspond to taking compositions of maps (cf. Proposition 3.13). So, we have to interpret $\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}},\dot{\boldsymbol{t}})$ as a coefficient in our algebraic setting. For this reason, fully pointed pre-generators generate the coefficient algebra (see §3.1). This complication in the definition repairs this issue and is also supplemented by the following observation: The algorithm underlying the definition of $\mathfrak{ev}$ can be applied not only for generators. For instance,

$$
\begin{aligned}
\mathfrak{ev}(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}},\dot{\boldsymbol{y}})) &= \#(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y})) && \in \mathbb{Z}_2 \\
\mathfrak{ev}(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot},\dot{\boldsymbol{y}})) &= \big(\boldsymbol{x}\longmapsto \#\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y})\big) && \in \mathrm{Hom}(\widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}),\mathbb{Z}_2) \\
\mathfrak{ev}(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}},\overset{\uparrow}{\boldsymbol{y}})) &= \#\big(\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\boldsymbol{x},\boldsymbol{y})\big)\cdot \boldsymbol{y} && \in \widehat{\mathrm{CF}}(\boldsymbol{\alpha},\boldsymbol{\beta}).
\end{aligned}
$$

Hence, fully pointed pre-generators naturally correspond to elements in the coefficient ring of Heegaard Floer homology.

Furthermore, this observation indicates that our setting provides a unified language for all elements of Floer homologies. As we will see in the next section, this will lead us to some kind of *reformulation* of Heegaard Floer homology in terms of our setting.

Example 5.1. We would like to calculate the symbol homology in a simple situation. Suppose we are given a Heegaard diagram $\mathcal{H} = (T^2,\boldsymbol{\alpha},\boldsymbol{\beta})$ where $\boldsymbol{\alpha} = \{\mu\}$ consists of a meridian μ and $\boldsymbol{\beta} = \{\lambda\}$ of a longitude λ such that $\#(\mu,\lambda) = 1$. Denote by $\boldsymbol{x}$ the intersection point of μ and λ. Now set $B = \{\boldsymbol{\alpha},\boldsymbol{\beta}\}$. Since $\mathcal{I}_B = \{[(\boldsymbol{\alpha},\boldsymbol{\beta})]\}$ is a one-point set we drop it from the notation of all moduli spaces. The fully pointed pre-generator $\mathcal{M}^0(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})$ is the only existing non-trivial fully pointed element. However, by definition $\mathcal{M}^0(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) = 1$ inside $\mathbb{F}$. Hence, $\mathbb{F} \cong \mathbb{Z}_2$ with $\partial_{\mathbb{F}} = 0$ such that $\mathfrak{f}_* \cong \mathbb{Z}_2$.

There is only one non-trivial generator, namely $X = \mathcal{M}^0(\overset{\downarrow}{\cdot},\overset{\uparrow}{\dot{\boldsymbol{x}}})$. Now, observe that $X^{\boxtimes 2} = \mathcal{M}^0(\overset{\downarrow}{\cdot},\dot{\boldsymbol{x}}) \times \mathcal{M}^0(\dot{\boldsymbol{x}},\overset{\uparrow}{\boldsymbol{x}})$ and that

$$
\begin{aligned}
X^{\boxtimes 3} &= \big(\mathcal{M}^0(\overset{\downarrow}{\cdot},\dot{\boldsymbol{x}}) \times \mathcal{M}^0(\dot{\boldsymbol{x}},\overset{\uparrow}{\dot{\boldsymbol{x}}})\big) \boxtimes \mathcal{M}^0(\overset{\downarrow}{\cdot},\overset{\uparrow}{\dot{\boldsymbol{x}}}) = \mathcal{M}^0(\overset{\downarrow}{\cdot},\dot{\boldsymbol{x}}) \times \big(\mathcal{M}^0(\dot{\boldsymbol{x}},\overset{\uparrow}{\dot{\boldsymbol{x}}}) \boxtimes \mathcal{M}^0(\overset{\downarrow}{\cdot},\overset{\uparrow}{\dot{\boldsymbol{x}}})\big) \\
&= \mathcal{M}^0(\overset{\downarrow}{\cdot},\dot{\boldsymbol{x}}) \times \big(\mathcal{M}^0(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) \bullet \mathcal{M}^0(\dot{\boldsymbol{x}},\overset{\uparrow}{\dot{\boldsymbol{x}}})\big) = \mathcal{M}^0(\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) \bullet \big(\mathcal{M}^0(\overset{\downarrow}{\cdot},\dot{\boldsymbol{x}}) \times \mathcal{M}^0(\dot{\boldsymbol{x}},\overset{\uparrow}{\dot{\boldsymbol{x}}})\big) \\
&= X^{\boxtimes 2}.
\end{aligned}
$$

This is the only existing relation and, hence, $\mathcal{S}$ is isomorphic to the associative $\mathbb{Z}_2$-algebra $\mathcal{R} \subset M_3(\mathbb{Z}_2)$ which is generated by the matrix

$$
A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
$$

The associated symbol homology can be written as $\mathfrak{sh}_* = \mathcal{R} \cup \{[\widehat{\Omega}]\}$ where $\mathfrak{sh}_* \backslash [\widehat{\Omega}]$ is isomorphic to $\mathcal{R}$ as a $\mathbb{Z}_2$-algebra.

Example 5.2. Suppose we are given a Heegaard diagram $\mathcal{H} = (T^2,\boldsymbol{\alpha},\boldsymbol{\beta})$ where both $\boldsymbol{\alpha} = \{\mu_1\}$ and $\boldsymbol{\beta} = \{\mu_2\}$ consist of a meridian such that μ_1 and μ_2 intersect in a canceling pair of intersection points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ where $\boldsymbol{x}_1$ denotes the one with higher relative grading.

The only non-trivial fully pointed pre-generator is $\widehat{\mathcal{M}}^1(\dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2)$. Thus, the coefficient algebra $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2[X]$. Furthermore, there are only three non-trivial generators, namely

$$X_1^0 = \mathcal{M}^0(\overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{x}_1}),\ X_2^0 = \mathcal{M}^0(\overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{x}_2})\ \text{and}\ X_2^1 = \widehat{\mathcal{M}}^1(\overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{x}_2}).$$

These fulfill the following relations

$$
\begin{array}{llllll}
(X_1^0)^{\boxtimes 3} & = & (X_1^0)^{\boxtimes 2} & X_1^0 \boxtimes X_2^0 & = & 0 \\
(X_2^0)^{\boxtimes 3} & = & (X_2^0)^{\boxtimes 2} & X_2^1 \boxtimes (X_2^0)^{\boxtimes 2} & = & X_2^1 \boxtimes X_2^0 \\
(X_2^1)^{\boxtimes 2} & = & 0 \quad \text{and} & (X_1^0)^{\boxtimes 2} \boxtimes X_2^1 & = & X_1^0 \boxtimes X_2^1 \\
X_2^1 \boxtimes X_1^0 & = & 0 & X_2^0 \boxtimes X_2^1 & = & 0 \\
X_2^0 \boxtimes X_1^0 & = & 0 & &
\end{array}
$$

So, $\mathcal{S}$ is isomorphic to the associative $\mathbb{Z}_2[X]$-algebra $\mathcal{R} \subset M_6(\mathbb{Z}_2)[X]$ which is generated by the matrices

$$B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},\ \ C = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\ \ \text{and}\ D,$$

where $D_{ij} = 1$ for (i, j) equal to $(1, 3)$, $(4, 4)$, $(4, 5)$, $(4, 6)$, $(5, 4)$, $(5, 5)$, $(5, 6)$ and $D_{ij} = 0$ otherwise. We see that $\mathfrak{sh}_* = \mathcal{R} \cup \{[\widehat{\Omega}]\}$ such that $\mathfrak{sh}_* \backslash \{[\widehat{\Omega}]\} \cong \mathcal{R}$ as $\mathbb{Z}_2[X]$-algebras.

To get an idea how the techniques from symbol homologies can be applied, we discuss the following two examples. In Example 5.3 we intend to illustrate the relationship between maps in Heegaard Floer homology and elements of the symbol algebra (see §3.3). In Example 5.4 we give an easy but explicitly worked out example for a transfer of a property from Heegaard Floer homology to knot Floer homology. We will apply the presented technique in §10 and §9 (cf. §1).

Example 5.3. Given a Heegaard triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$, denote by $\mathcal{H}_1$ the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, denote by $\mathcal{H}_2$ the Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma})$ and denote by $\mathcal{H}_3$ the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma})$. We define a map

$$\widehat{F}_{\boldsymbol{\alpha}, \boldsymbol{\beta}\boldsymbol{\gamma}}\colon\ \widehat{\mathrm{CF}}(\mathcal{H}_1) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_2) \longrightarrow \widehat{\mathrm{CF}}(\mathcal{H}_3)$$

in the following way: Let $B = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ and denote by B' the element $[(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta})]$ of $\mathcal{I}_B$. For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\beta}} \cap \mathbb{T}_{\boldsymbol{\gamma}}$, we define

$$\widehat{F}_{\boldsymbol{\alpha}, \boldsymbol{\beta}\boldsymbol{\gamma}}(\boldsymbol{x} \otimes \boldsymbol{y}) = \sum_{\boldsymbol{q} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}} \#\mathcal{M}_{B'}^0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{q}) \cdot \boldsymbol{q}$$

and extend it to $\widehat{\mathrm{CF}}(\mathcal{H}_1) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_2)$ as a bilinear map. We interpret the moduli space $\mathcal{M}_{B'}^0(\cdot, \cdot, \boldsymbol{q})$ as a generator, by attaching data to its vertices. Instead of using the notations from the previous sections, we indicate the decorations like introduced at the beginning of this section: we write $\mathcal{M}_{B'}^0(\overset{\downarrow}{\cdot}, \overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{q}})$ for the generator $(\mathcal{M}_{B'}^0)_{(P^\bullet, F^\downarrow, F^\uparrow)}$, where $P^\bullet = \emptyset$, $F^\downarrow = \{(\boldsymbol{\beta}, \boldsymbol{\alpha}), (\boldsymbol{\gamma}, \boldsymbol{\beta})\}$ and $F^\uparrow = \{\{(\boldsymbol{\alpha}, \boldsymbol{\gamma}), \boldsymbol{q}\}\}$. Now consider the following element of the symbol algebra:

$$\mathfrak{s} = \bigboxplus_{\boldsymbol{q} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}} \mathcal{M}_{B'}^0(\overset{\downarrow}{\cdot}, \overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{q}}).$$

This symbol induces an element in the symbol homology, since all moduli spaces in the $\boxplus$-sum are 0-dimensional and it is easy to see that $\mathfrak{ev}(\mathfrak{s}) = \widehat{F}_{\alpha,\beta\gamma}$. Analogously, consider the following element

$$\mathfrak{s}_{\alpha\beta} = \coprod_{\boldsymbol{y}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\beta}} \widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}})$$

for which $\mathfrak{ev}(\mathfrak{s}_{\alpha\beta}) = \widehat{\partial}_{\mathcal{H}_1}$ holds. In a similar vein, we define elements $\mathfrak{s}_{\beta\gamma}$ and $\mathfrak{s}_{\alpha\gamma}$ with the properties that $\mathfrak{ev}(\mathfrak{s}_{\beta\gamma}) = \widehat{\partial}_{\mathcal{H}_2}$ and $\mathfrak{ev}(\widehat{\partial}_{\alpha\gamma}) = \widehat{\partial}_{\mathcal{H}_3}$. All these elements induce classes in $\mathfrak{sh}_*$ we denote by $\mathfrak{s}_{\alpha\beta;*}$, $\mathfrak{s}_{\beta\gamma;*}$, $\mathfrak{s}_{\alpha\gamma;*}$ and $\mathfrak{s}_*$. We would like to prove that $\mathfrak{q}_* = [\mathfrak{q}]$ with $\mathfrak{q} = \mathfrak{s} \boxtimes \mathfrak{s}_{\alpha\gamma} \boxplus \mathfrak{s}_{\alpha\beta} \boxtimes \mathfrak{s} \boxplus \mathfrak{s}_{\beta\gamma} \boxtimes \mathfrak{s}$ vanishes in $\mathfrak{sh}_*$. In fact,

$$\partial_{\mathfrak{sh}}\Big(\coprod_{\boldsymbol{y}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\gamma}} \mathcal{M}^1_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}},) \Big) = \coprod_{\boldsymbol{y}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\gamma}} \partial_{\mathfrak{sh}}\big(\mathcal{M}^1_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}},) \big),$$

where $\partial_{\mathfrak{sh}}\big(\mathcal{M}^1_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}) \big)$ equals

$$\coprod_{\boldsymbol{x}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\beta}} \widehat{\mathcal{M}}^1_{\alpha\beta}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{x}}) \boxtimes \mathcal{M}^0_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}) \quad \boxplus \quad \coprod_{\boldsymbol{q}\in\mathbb{T}_{\beta}\cap\mathbb{T}_{\gamma}} \widehat{\mathcal{M}}^1_{\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{q}}) \boxtimes \mathcal{M}^0_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}) \quad \boxplus$$
$$\coprod_{\boldsymbol{r}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\gamma}} \mathcal{M}^0_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{r}}) \boxtimes \widehat{\mathcal{M}}^1_{\alpha\gamma}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}).$$

Hence, $\partial_{\mathfrak{sh}}\Big(\coprod_{\boldsymbol{y}\in\mathbb{T}_{\alpha}\cap\mathbb{T}_{\gamma}} \mathcal{M}^1_{\alpha\beta\gamma}(\overset{\downarrow}{\cdot},\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}},) \Big) = \mathfrak{q}$ which implies $\mathfrak{q}_* = 0$. Thus, we have

$$\begin{aligned}
0 = \mathfrak{ev}_*(\mathfrak{q}_*) &= \mathfrak{ev}_*(\mathfrak{s}_{\alpha\gamma;*}) \,\widehat{\circ}\, \mathfrak{ev}_*(\mathfrak{s}_*) &+& \quad \mathfrak{ev}_*(\mathfrak{s}_*) \,\widehat{\circ}\, \mathfrak{ev}_*(\mathfrak{s}_{\alpha\beta;*}) \\
&&+& \quad \mathfrak{ev}_*(\mathfrak{s}_*) \,\widehat{\circ}\, \mathfrak{ev}_*(\mathfrak{s}_{\beta\gamma;*}) \\[2mm]
&= \quad \widehat{\partial}_{\mathcal{H}_3} \circ \widehat{F}_{\alpha,\beta\gamma} &+& \quad \widehat{F}_{\alpha,\beta\gamma} \circ \big(\widehat{\partial}_{\mathcal{H}_1} \otimes \mathrm{id}_{\beta\gamma} \big) \\
&&+& \quad \widehat{F}_{\alpha,\beta\gamma} \circ \big(\mathrm{id}_{\alpha\beta} \otimes \widehat{\partial}_{\mathcal{H}_2} \big).
\end{aligned}$$

Consequently, $\widehat{F}_{\alpha,\beta\gamma}$ is a chain map. This illustrates that maps between Floer theories can be expressed as elements in the symbol homology and that properties of these maps are encoded in the image of $\partial_{\mathfrak{sh}}$. In this way, a property of a map is equivalent to the vanishing of a suitable obstruction class in the symbol homology.

Example 5.4. We point the reader to Example 5.3 for the notations and definitions used here. We have seen in Example 5.3 that $\widehat{F}_{\alpha,\beta\gamma}$ is a chain map from $\widehat{\mathrm{CF}}(\mathcal{H}_1) \otimes \widehat{\mathrm{CF}}(\mathcal{H}_2)$ to $\widehat{\mathrm{CF}}(\mathcal{H}_3)$. We denoted by $\mathfrak{s}$ its canonical symbol (see §3.3). We proved that $\widehat{F}_{\alpha,\beta\gamma}$ is a chain map by showing that $\mathfrak{q}_* = [\mathfrak{q}]$ vanishes in homology. However, obverse that $\mathfrak{q}_* = P(\mathfrak{s}_*)$ for

$$P(X) = X \boxtimes \mathfrak{s}_{\alpha\gamma;*} \boxplus \mathfrak{s}_{\alpha\beta;*} \boxtimes X \boxplus \mathfrak{s}_{\beta\gamma;*} \boxtimes X.$$

Hence, the polynomial $P(X) \in \mathbb{P}(\mathcal{G})$ encodes the chain map property. Applying Theorem 4.3, we see that $\mathfrak{F}_*(\mathfrak{s}_*)$ fulfills the property $P_{\mathfrak{F}}$, i.e. the equality

$$(5.1) \qquad 0 = P_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s}_*)) = \mathfrak{F}_*(\mathfrak{s}_*) \boxtimes \mathfrak{F}_*(\mathfrak{s}_{\alpha\gamma;*}) \boxplus \mathfrak{F}_*(\mathfrak{s}_{\alpha\beta;*}) \boxtimes \mathfrak{F}_*(\mathfrak{s}_*) \boxplus \mathfrak{F}_*(\mathfrak{s}_{\beta\gamma;*}) \boxtimes \mathfrak{F}_*(\mathfrak{s}_*)$$

holds. Now observe that $\mathfrak{F}_*(\mathfrak{s}_{\boldsymbol{\alpha\gamma};*})$, $\mathfrak{F}_*(\mathfrak{s}_{\boldsymbol{\alpha\beta};*})$ and $\mathfrak{F}_*(\mathfrak{s}_{\boldsymbol{\beta\gamma};*})$ are the canonical symbols of $\partial_{\mathcal{H}_3}^{\bullet;\bullet}$, $\widehat{\partial}_{\mathcal{H}_1}^{\bullet;\bullet}$ and $\widehat{\partial}_{\mathcal{H}_2}^{\bullet;\bullet}$, respectively (cf. §2.2). By Proposition 3.13, equation (5.1) implies

$$
\begin{aligned}
0 \;&= \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) \boxtimes \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{\boldsymbol{\alpha\gamma};*})) &&\boxplus\;\; \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{\boldsymbol{\alpha\beta};*})) \boxtimes \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) \\
&&&\boxplus\;\; \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{\boldsymbol{\beta\gamma};*})) \boxtimes \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) \\
&= \partial_{\mathcal{H}_3}^{\bullet;\bullet} \circ \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) &&+\;\; \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) \circ (\widehat{\partial}_{\mathcal{H}_1}^{\bullet;\bullet} \otimes \mathrm{id}_{\boldsymbol{\beta\gamma}}) \\
&&&+\;\; \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) \circ (\mathrm{id}_{\boldsymbol{\alpha\beta}} \otimes \widehat{\partial}_{\mathcal{H}_2}^{\bullet;\bullet}).
\end{aligned}
$$

Hence,

$$
\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_*)) \colon \widehat{\mathrm{CFK}}(\mathcal{H}_1) \otimes \widehat{\mathrm{CFK}}(\mathcal{H}_2) \longrightarrow \widehat{\mathrm{CFK}}(\mathcal{H}_3)
$$

is a chain map. But, observe that it equals $F_{\boldsymbol{\alpha},\boldsymbol{\beta\gamma}}^{\bullet;\bullet}$ from §9. As we see, the chain map property of $F_{\boldsymbol{\alpha},\boldsymbol{\beta\gamma}}^{\bullet;\bullet}$ is a consequence of the chain map property of $\widehat{F}_{\boldsymbol{\alpha},\boldsymbol{\beta\gamma}}$ and Theorem 4.3.

The technique presented in Example 5.4 is used in §9 to give an invariance proof for cobordism maps in knot Floer homology and for a surgery exact triangle in §10 (see Theorem 10.2).

6. Recovering Heegaard Floer Theory from Symbol Homology

6.1. **Homology.** Consider in $(\mathcal{T}, +, \times)$ the $\mathbb{F}$-subalgebra C generated by pre-generators for which $\#(F^{\downarrow}) = 0$ and $\#(F^{\uparrow}) = 1$ holds. The subsemialgebra $\mathcal{X}^1 = \Phi(C)$ of $(\widehat{\mathcal{T}}, \boxplus, \boxtimes)$ is naturally equipped with a differential $\partial_{\mathfrak{sh}} \colon \mathcal{X}^1 \longrightarrow \mathcal{X}^1$ by applying the algorithm presented in §3.2. Denote by $\mathcal{X}_* = H_*(\mathcal{X}^1, \partial_{\mathfrak{sh}})$ the associated homology theory. Given an element $[(\boldsymbol{\alpha},\boldsymbol{\beta})] \in \mathcal{I}_B$, we consider

$$
\mathcal{X}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0 = \mathfrak{f}_* \big\langle [\mathcal{A}] = [A_{(P^\bullet, F^{\downarrow}, F^{\uparrow})}] \in \mathcal{X}_* \,|\, A = \mathcal{M}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0, \#(F^{\downarrow}) = 0, \#(F^{\uparrow}) = 1 \big\rangle
$$

which is a $\mathfrak{f}_*$-submodule of $\mathcal{X}_*$ and consider

$$
\mathbb{O}_{(\boldsymbol{\alpha},\boldsymbol{\beta})} = \bigboxplus_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \big[\mathcal{M}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0 (\overset{\downarrow}{\cdot}, \overset{\uparrow}{\boldsymbol{y}}) \big]
$$

which is an element in the symbol homology $\mathfrak{sh}_*$. Denote by $\mathcal{H}$ the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and denote by $\mathfrak{s}_{\boldsymbol{\alpha\beta}}$ the canonical symbol (see §3.3) of $\widehat{\partial}_{\mathcal{H}}$. Then we obtain the following interpretation of Heegaard Floer homology.

Theorem 6.1. *Multiplication from the right with the element $\mathfrak{s}_{\boldsymbol{\alpha\beta}} \boxtimes \mathbb{O}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ defines a differential $\partial_{\mathcal{X}}$ on $\mathcal{X}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0$. Denote by $\big(\mathcal{X}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0\big)_*$ the induced homology theory.*

(i) *We have that*

(6.1)
$$
\mathcal{X}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0 \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 \cong \widehat{\mathrm{CF}}(\mathcal{H}),
$$

where the left side is equipped with the differential $\partial_{\mathcal{X}} \otimes \mathrm{id}$ and where we equip $\mathbb{Z}_2$ with the structure of a $\mathfrak{f}_*$-module using the map $(\mathfrak{ct}_{\mathbb{F}})_*$ (see (3.11)). Furthermore,

(6.2)
$$
\big(\mathcal{X}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0\big)_* \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 = H_*(\mathcal{X}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}^0 \otimes_{\mathfrak{f}_*} \mathbb{Z}_2, \partial_{\mathcal{X}} \otimes \mathrm{id}) \cong \widehat{\mathrm{HF}}(\mathcal{H}),
$$

where $\mathbb{Z}_2$ carries the structure of a $\mathfrak{f}_*$-module.

(ii) *Suppose we are given a map*

$$F \colon \bigotimes_{i=1}^{n-1} \widehat{\mathrm{CF}}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_{i+1}) \longrightarrow \widehat{\mathrm{CF}}(\boldsymbol{\alpha}_n, \boldsymbol{\alpha}_1)$$

with canonical symbol $\mathfrak{s}_F$. Multiplication from the right with the symbol $\mathfrak{s}_F \boxtimes \mathbb{O}_{(\boldsymbol{\alpha}_n, \boldsymbol{\alpha}_1)}$ defines a map

$$\cdot \boxtimes \mathfrak{s}_F \boxtimes \mathbb{O}_{(\boldsymbol{\alpha}_n, \boldsymbol{\alpha}_1)} \colon \Big(\bigotimes_{i=1}^{n-1} \mathcal{X}^0_{(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_{i+1})} \Big)_* \longrightarrow \big(\mathcal{X}^0_{(\boldsymbol{\alpha}_n, \boldsymbol{\alpha}_1)} \big)_*$$

such that, under the isomorphism given in part (i), this map corresponds to F (even on the chain level).

Proof. Denote by $\mathcal{H}$ the Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ denote by $X_{\boldsymbol{x}}$ the element $\big[\mathcal{M}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(\cdot, \overset{\uparrow}{\boldsymbol{x}}) \big] \otimes 1$. Define $\rho(X_{\boldsymbol{x}}) = \boldsymbol{x}$ and extend to a map

$$\rho \colon \mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 \longrightarrow \widehat{\mathrm{CF}}(\mathcal{H})$$

as a morphism of modules, where

(6.3) $$\rho(f \bullet \mathcal{A} \otimes 1) = \mathfrak{ct}_{\mathbb{F}}(f) \cdot \rho(\mathcal{A} \otimes 1)$$

for $\mathcal{A} \in \mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ and $f \in \mathfrak{f}_*$. It is easy to see that ρ is a bijection. Furthermore, consider the following chain of equalities:

$$
\begin{aligned}
(\rho \circ \partial_{\mathcal{X}} \otimes \mathrm{id})(X_{\boldsymbol{x}}) &= \rho \Big(\bigboxplus_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \big[\widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}}) \big] \bullet X_{\boldsymbol{y}} \Big) \\
&= \sum_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \# \widehat{\mathcal{M}}^1_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{y} \\
&= \widehat{\partial}_{\mathcal{H}}(\boldsymbol{x}) \\
&= \widehat{\partial}_{\mathcal{H}} \big(\rho(X) \big).
\end{aligned}
$$

Since the map ρ is a bijection, the given computation shows that ρ induces a map

$$\rho_* \colon H_*(\mathcal{X}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2, \partial_{\mathcal{X}} \otimes \mathrm{id}) \longrightarrow \widehat{\mathrm{HF}}(\mathcal{H})$$

which is an isomorphism. This proves that the right equality in (6.2) is true. The left equality in (6.2) is clear.

Given a map $F \colon \widehat{\mathrm{CF}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \longrightarrow \widehat{\mathrm{CF}}(\boldsymbol{\alpha}', \boldsymbol{\beta}')$ which is defined by counting elements of moduli spaces of Whitney polygons, there are generators $\mathcal{R}_{i,j}$, $j = 1, \ldots, l$ and $i = 1, \ldots, k_j$ such that

$$\mathfrak{s}_F = \bigboxplus_{j=1}^{l} \bigboxtimes_{i=1}^{k_j} [\mathcal{R}_{i,j}]$$

is its canonical symbol. The data of $\mathcal{R}_{i,j}$ are denoted by $F^{\downarrow}(i,j)$, $F^{\uparrow}(i,j)$ and $P^{\bullet}(i,j)$. Since $\mathfrak{s}_F$ is the canonical symbol of F, for every i and j the flow-in vertices of $\mathcal{R}_{i,j}$ appear

as flow-out vertices of $\mathcal{R}_{i-1,j}$. We denote by $F^{\downarrow}$, $F^{\uparrow}$ and $P^{\bullet}$ the decorations of $\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \overset{\uparrow}{\dot{\boldsymbol{x}}})$ and we denote by $I_{\boldsymbol{y}}$ the set of j for which $\{(\boldsymbol{\alpha}', \boldsymbol{\beta}'), \boldsymbol{y}\} \in P^{\bullet}(\mathcal{R}_{k_j,j})$. Hence, we have that

$$[\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \overset{\uparrow}{\dot{\boldsymbol{x}}})] \boxtimes \mathfrak{s}_F \boxtimes \mathbb{O} = \bigboxplus_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}'} \cap \mathbb{T}_{\boldsymbol{\beta}'}} Q_{\boldsymbol{y}} \bullet [\mathcal{M}^0_{(\boldsymbol{\alpha}',\boldsymbol{\beta}')}(\dot{\boldsymbol{y}}, \overset{\uparrow}{\dot{\boldsymbol{y}}})]$$

with

$$
\begin{aligned}
Q_{\boldsymbol{y}} &= \bigboxplus_{j \in I_{\boldsymbol{y}}} \Big(\big[\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \bullet \mathcal{K}^{\downarrow,F^{\uparrow}}(\mathcal{K}^{\uparrow}(\mathcal{R}_{1,j})) \bullet \big(\overset{k_j}{\underset{i=2}{\bigbullet}} \mathcal{K}^{\downarrow,F^{\uparrow}(i-1,j)}(\mathcal{K}^{\uparrow}(\mathcal{R}_{i,j})) \big) \Big) \\
&= \bigboxplus_{j \in I_{\boldsymbol{y}}} \Big(\big[\mathcal{K}^{\downarrow,F^{\uparrow}}(\mathcal{K}^{\uparrow}(\mathcal{R}_{1,j})) \bullet \big(\overset{k_j}{\underset{i=2}{\bigbullet}} \mathcal{K}^{\downarrow,F^{\uparrow}(i-1,j)}(\mathcal{K}^{\uparrow}(\mathcal{R}_{i,j})) \big) \big] \Big),
\end{aligned}
$$

where the second equality holds since in the coefficient algebra $\mathcal{M}^0_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1$. But it is not hard to see that $\mathfrak{ct}_{\mathbb{F}}(Q_{\boldsymbol{y}}) = F(\boldsymbol{x})|_{\boldsymbol{y}}$. So, under the morphism ρ the multiplication from the right with $\mathfrak{s}_F \boxtimes \mathbb{O}$ corresponds to the map F. $\square$

For every element $\mathfrak{s} \in \mathfrak{sh}_*$, there exists a unique pair of attaching circles $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ such that $\mathfrak{s} \boxtimes \mathbb{O}_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \neq \widehat{\Omega}$. Hence, in products we can suppress the attaching circles from the notation and just write $\mathfrak{s} \boxtimes \mathbb{O}$ instead.

Definition 6.2. For a symbol $\mathfrak{s} \in \mathfrak{sh}_*$, define $\mathfrak{s}_{\mathbb{O}}$ to be the product $\mathfrak{s} \boxtimes \mathbb{O}$. Furthermore, for an element $P \in \mathbb{P}$ we define $P_{\mathbb{O}}$ to be the polynomial expression we obtain by replacing all coefficients c_i by $c_i \boxtimes \mathbb{O}$.

Proposition 6.3. *A symbol $\mathfrak{s}_* \in \mathfrak{sh}_*$ fulfills a property P if and only if $\mathfrak{s}_{\mathbb{O}}$ fulfills the property $P_{\mathbb{O}}$.*

Proof. For symbols $\mathfrak{s}_* = [\mathfrak{s}]$, $\mathfrak{t}_* = [\mathfrak{t}] \in \mathfrak{sh}_*$ a simple calculation shows that $\mathfrak{s} \boxtimes \mathbb{O} \boxtimes \mathfrak{t} = \mathfrak{s} \boxtimes \mathfrak{t}$. Consequently, we have that

$$P_{\mathbb{O}}(\mathfrak{s}_{\mathbb{O}}) = P_{\mathbb{O}}(\mathfrak{s}_* \boxtimes \mathbb{O}) = P(\mathfrak{s}_*) \boxtimes \mathbb{O}.$$

Now suppose that $P(\mathfrak{s}_*)$ vanishes, then $P_{\mathbb{O}}(\mathfrak{s}_* \boxtimes \mathbb{O}) = 0$. Conversely, given that $P_{\mathbb{O}}((\mathfrak{s}_*)_{\mathbb{O}})$ vanishes, we have that $P(\mathfrak{s}_*) \boxtimes \mathbb{O} = 0$. So, there is an element $\mathfrak{q}$ such that $P(\mathfrak{s}) \boxtimes \mathbb{O} = \partial_{\mathfrak{sh}}(\mathfrak{q})$. Since $\mathbb{O}$ is not an element in the image of $\partial_{\mathfrak{sh}}$, the element $\mathfrak{q}$ is of the form $\mathfrak{t} \boxtimes \mathbb{O}$ which implies that $P(\mathfrak{s}) = \partial_{\mathfrak{sh}}(\mathfrak{t})$ and, hence, $P(\mathfrak{s}_*) = 0$. $\square$

Proof of Theorem 1.1. Recall that by Theorem 6.1 we may regard the modules $\mathcal{X}^0_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ and their respective homology theories $(\mathcal{X}^0_{(\boldsymbol{\alpha},\boldsymbol{\beta})})_*$ as being equivalent to Heegaard Floer homology. Furthermore, symbols of type $\mathfrak{s}_{\mathbb{O}}$ can be regarded as maps between Floer homologies. Applying Theorem 4.3 and the fact that *filtering* induces a morphism $\mathfrak{F}_*$, we see that if a map $\mathfrak{s}_{\mathbb{O}}$ fulfills a property $P_{\mathbb{O}}$, then $\mathfrak{F}_*(\mathfrak{s}_{\mathbb{O}}) = \mathfrak{F}_*(\mathfrak{s})_{\mathbb{O}}$ fulfills the property $(P_{\mathfrak{F}})_{\mathbb{O}}$. $\square$

6.2. **Cohomology.** In a similar vein it is possible to recover Heegaard Floer cohomology in terms of the symbol homology theory. To do that, we have to give a couple of definitions: Consider in $(\mathcal{T}, +, \times)$ the $\mathbb{F}$-subalgebra C generated by pre-generators with the property that $\#(F^\downarrow) > 0$ and $\#(F^\uparrow) \le 1$. The semialgebra $\mathcal{X}^{coh} = \Phi(C)$ of $(\widehat{\mathcal{T}}, \boxplus, \boxtimes)$ is naturally equipped with a differential $\partial_{\mathfrak{sh}} \colon \mathcal{X}^{coh} \longrightarrow \mathcal{X}^{coh}$ by applying the algorithm presented in §3.2. Denote by $\mathcal{X}^{coh}_*$ the homology theory associated to the complex $(\mathcal{X}^{coh}, \partial_{\mathfrak{sh}})$. Given an element $[(\boldsymbol{\alpha}, \boldsymbol{\beta})] \in \mathcal{I}_B$, consider

$$\mathcal{X}^{coh}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} = \mathfrak{f}_* \big\langle [\mathcal{A}] = [A_{(P^\bullet, F^\downarrow, F^\uparrow)}] \in \mathcal{X}^{coh}_* \,|\, A = \mathcal{M}^0_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}, \#(F^\downarrow) = 1, \#(F^\uparrow) = 0 \big\rangle.$$

Define $\mathcal{H}$, $\mathfrak{s}_{\boldsymbol{\alpha\beta}}$ and $\mathbb{O}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$ as in §6.1.

Theorem 6.4. *Multiplication from the left with the element $\mathbb{O}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \boxtimes \mathfrak{s}_{\boldsymbol{\alpha\beta}}$ defines a differential $\partial^{coh}_{\mathcal{X}}$ on $\mathcal{X}^{coh}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}$. Denote by $(\mathcal{X}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})})^{coh}_*$ the induced homology theory.*

(i) *We have that*

$$(6.4) \qquad \mathcal{X}^{coh}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 \cong \widehat{\mathrm{CF}}^*(\mathcal{H})$$

where the left side is equipped with the differential $\partial^{coh}_{\mathcal{X}} \otimes \mathrm{id}$ and where we equip $\mathbb{Z}_2$ with the structure of an $\mathfrak{f}_*$-module using the map $(\mathfrak{ct}_{\mathbb{F}})_*$ (see (3.11)). Furthermore,

$$(6.5) \qquad \big(\mathcal{X}^{coh}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}\big)_* \otimes_{\mathfrak{f}_*} \mathbb{Z}_2 = H_*(\mathcal{X}^{coh}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \otimes_{\mathfrak{f}_*} \mathbb{Z}_2, \partial^{coh}_{\mathcal{X}} \otimes \mathrm{id}) \cong \widehat{\mathrm{HF}}^*(\mathcal{H}).$$

where $\mathbb{Z}_2$ carries the structure of a $\mathfrak{f}_*$-module.

(ii) *Given a map between two Heegaard Floer chain complexes $\widehat{\mathrm{CF}}(\boldsymbol{\beta}, \boldsymbol{\alpha})$ and $\widehat{\mathrm{CF}}(\boldsymbol{\beta}', \boldsymbol{\alpha}')$, denote by $\mathfrak{s}_F$ its associated canonical symbol. Then, multiplication from the left with the element $\mathbb{O}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \boxtimes \mathfrak{s}_F$ induces a map*

$$\mathbb{O}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})} \boxtimes \mathfrak{s}_F \boxtimes \cdot \colon \big(\mathcal{X}^{coh}_{(\boldsymbol{\alpha}, \boldsymbol{\beta})}\big)_* \longrightarrow \big(\mathcal{X}^{coh}_{(\boldsymbol{\alpha}', \boldsymbol{\beta}')}\big)_*$$

such that, under the isomorphism given in part (i), this map corresponds to F (even on the chain level).

Proof. The proof goes the same way as the proof of Theorem 6.1. $\qquad\qquad\square$

7. U-equivariant Symbol Homology

The construction of symbol homology and its ambient algebra can be altered in various ways without destroying the properties derived in §3. Here, we present a variant of this theory by entering a U-variable. This modification is necessary to capture information from flavors of Heegaard Floer homology which also admit a U-variable in their definition. For a moduli space $\mathcal{M}^\mu_{B'}$, we define

$$\mathcal{M}^{\mu;i}_{B'} = \{\phi \in \mathcal{M}^\mu_{B'} \,|\, n_w(\phi) = i\}.$$

In the following, we will decorate the $\mathcal{M}^{\mu;i}_{B'}$ with data and follow the construction process as outlined in §3 with some slight adaptions: Instead of $\mathbb{F}$ as coefficients we use $\mathbb{F}[U]$ as coefficients to generate the algebra $\widehat{\mathcal{T}}$. Furthermore, we define the sum $\boxplus$ and the product $\boxtimes$ as in §3, with the additional condition

$$(U \bullet \mathcal{A}) \boxplus (U \bullet \mathcal{B}) = U \bullet (\mathcal{A} \boxplus \mathcal{B})$$

and

$$(U \bullet \mathcal{A}) \boxtimes \mathcal{B} = U \bullet (\mathcal{A} \boxtimes \mathcal{B}) = \mathcal{A} \boxtimes (U \bullet \mathcal{B})$$

imposed. The symbol algebra we obtain with this new construction will be denoted by $\widehat{\mathcal{S}}_U$ and the associated symbol homology by $\mathfrak{sh}_{U,*}$. Every moduli space $\mathcal{M}_{B'}^{\mu}$ is a disjoint union of the $\mathcal{M}_{B'}^{\mu;i}$, for $i \geq 0$. Inspired by this, it is possible to define a map from the symbol algebra $\widehat{\mathcal{S}}$ to $\widehat{\mathcal{S}}_U$ in the following way: Given a generator $\mathcal{A} = (\mathcal{M}_{B'}^{\mu})_{(P^{\bullet}, F^{\downarrow}, F^{\uparrow})}$, we denote by $\mathcal{A}^i$ the element $(\mathcal{M}_{B'}^{\mu;i})_{(P^{\bullet}, F^{\downarrow}, F^{\uparrow})}$. We require that $\mathfrak{F}_U(\widehat{\Omega}) = \widehat{\Omega}$ and that for a generator $\mathcal{A}$ we have

$$\mathfrak{F}_U(\mathcal{A}) = \coprod_i U^i \bullet \mathcal{A}^i.$$

Observe that the sum is finite. This assignment extends to a morphism

$$\mathfrak{F}_U \colon \widehat{\mathcal{S}} \longrightarrow \widehat{\mathcal{S}}_U$$

of semialgebras.

Proposition 7.1. *If $\mathcal{J}_s$ is w-respectful, then the map $\mathfrak{F}_U$ is a chain map with respect to the differential $\partial_{\mathfrak{sh}}$ and, thus, descends to a $(\mathfrak{f}_*, \mathfrak{f}_{U,*})$-morphism*

$$\mathfrak{F}_{U;*} \colon \mathfrak{sh}_* \longrightarrow \mathfrak{sh}_{U,*}$$

of semialgebras.

Proof. The statement that $\mathfrak{F}_U$ is a chain map follows from the fact that intersection numbers are homotopical invariants and behave additive under splicing. The fact, that the map $\mathfrak{F}_{U;*}$ is a morphism follows from the fact that $\mathfrak{F}_U$ is a morphism which in turn is true by its definition. $\qquad\square$

Theorem 7.2. *If a map $\mathfrak{s}$ from a tensor product of Heegaard Floer chain complexes to another Heegaard Floer chain complex fulfills a property P, then the map $\mathfrak{F}_{U;*}(\mathfrak{s})$ between the corresponding $\mathrm{CFK}^{\bullet,-}$-knot Floer chain complexes fulfills the property $P_{\mathfrak{F}_U}$.*

Proof. The proof of this theorem follows the same lines as the proof of Theorem 1.1. $\qquad\square$

Although we did not write this down explicitly, there is a U-equivariant version of Theorem 6.1 (and Theorem 6.4) giving a model for $\mathrm{HFK}^{\bullet,-}$ in terms of the symbol homology theory.

8. Perturbed Symbol Homologies

In this section we briefly sketch a necessary extension of the symbol homology theories. Observe that the isomorphisms between Floer theories that are induced by – for instance – isotopies or perturbations of the path of almost complex structures use dynamic boundary conditions. Hence, these maps cannot be presented as elements in the symbol homology we defined in §3. However, we can extend the symbol homology theory by including moduli spaces with dynamic boundary conditions to the set of generators. The constructions given in §3 and §4 then carry over verbatim. Because of the similarity of these approaches, we will just specify which moduli spaces we have to include into our considerations.

8.1. **Perturbations of the Almost Complex Structure.** Given a perturbation $\mathcal{J}_{s,t}$ of the path $\mathcal{J}_{s,0}$, we additionally fix a homotopy $\mathcal{J}_{s,t}(\tau)$ where $\mathcal{J}_{s,t}(0) = \mathcal{J}_{s,t} * \mathcal{J}_{s,1-t}$ and $\mathcal{J}_{s,t}(1) = \mathcal{J}_{s,0}$. For a given set of attaching circles B we consider the following moduli spaces.

(1) For every $B' \in \mathcal{I}_B$ we consider $\mathcal{M}^{\mu}_{B';\mathcal{J}_{s,0}}$ with restrictions to the Maslov-index as given in part (1) of Definition 3.1.

(2) For every $B' \in \mathcal{I}_B$, we consider $\mathcal{M}^{\mu}_{B';\mathcal{J}_{s,1}}$ with restrictions to the Maslov-index as given in part (1) of Definition 3.1.

(3) For every element $B' \in \mathcal{I}_B$, we consider $\mathcal{M}^{i}_{B';\mathcal{J}_{s,t}}$, for $i = 0, 1$, which is the set of $\mathcal{J}_{s,t}$-holomorphic Whitney polygons with boundary conditions specified by B' with Maslov index i (see [4] or cf. [10]). Furthermore, we consider $\mathcal{M}^{i}_{B';\mathcal{J}_{s,1-t}}$, $i = -1, 0$, and

$$\mathcal{M}^{i}_{B';\tau} = \bigcup_{\tau \in [0,1]} \mathcal{M}^{i}_{B';\mathcal{J}_{s,t}(\tau)}$$

for $i = -1, 0$.

The constructions given in §3 carry over verbatim to provide a symbol homology theory with the following slight adaptions: We need to define $\partial_{\mathfrak{sh}}$ for the generators we obtain from (3). The algorithm presented in §3.2 applies here, as well, to provide a definition of $\partial_{\mathfrak{sh}}$ for all generators given by moduli spaces that come from (1), (2) and the spaces $\mathcal{M}^{i}_{B';\mathcal{J}_{s,t}}$. For $\mathcal{M}^{0}_{B';\tau}$ the following algorithm applies: There are two types of ends, the broken ends and the ends coming from $\tau \to 0/1$ which are $\mathcal{M}^{0}_{B';\mathcal{J}_{s,t}(0)}$ and $\mathcal{M}^{0}_{B';\mathcal{J}_{s,t}(1)}$. To the first type, i.e. the broken ends, the algorithm from §3.2 applies. To the second type we apply the following procedure: There is a canonical one-to-one correspondence between the vertices of $\mathcal{M}^{0}_{B';\tau}$ and the vertices of $\mathcal{M}^{0}_{B';\mathcal{J}_{s,t}(0)}$ (or $\mathcal{M}^{0}_{B';\mathcal{J}_{s,t}(1)}$). Hence, we may take the decorations of the vertices of $\mathcal{M}^{0}_{B';\tau}$ and attach them to the vertices of $\mathcal{M}^{0}_{B';\mathcal{J}_{s,t}(0)}$ (and $\mathcal{M}^{0}_{B';\mathcal{J}_{s,t}(1)}$). Analogously, we get a definition of $\partial_{\mathbb{F}}$ for fully pointed pre-generators coming from moduli spaces of (3).

Example 8.1. For $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ we have

$$
\begin{aligned}
\partial_{\mathfrak{sh}}\left(\mathcal{M}^{0}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\tau}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}})\right) \;=\; & \mathcal{M}^{0}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\mathcal{J}_{s,t}(0)}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}) \boxplus \mathcal{M}^{0}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\mathcal{J}_{s,t}(1)}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}) \\
& \boxplus \bigboxplus_{\boldsymbol{q} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \mathcal{M}^{0}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\mathcal{J}_{s,t}(\tau)}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{q}}) \boxtimes \widehat{\mathcal{M}}^{1}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\mathcal{J}_{s,t}(1)}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}}) \\
& \boxplus \bigboxplus_{\boldsymbol{q} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \widehat{\mathcal{M}}^{1}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\mathcal{J}_{s,t}(0)}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{q}}) \boxtimes \mathcal{M}^{0}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\mathcal{J}_{s,t}(\tau)}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}})
\end{aligned}
$$

For the notation of the decorations we point the reader to §5. The boundaries in lines 2 and 3 are given by the algorithm presented in §3.2. The first line is defined by the procedure presented above: we move the data from $\mathcal{M}^{0}_{(\boldsymbol{\alpha},\boldsymbol{\beta});\tau}(\overset{\downarrow}{\cdot},\overset{\uparrow}{\boldsymbol{y}})$ to its boundary components.

8.2. **Isotopies.** This is done precisely the same way as the case of perturbations of the almost complex structure. For a given Hamiltonian isotopy Φ we include the moduli spaces with dynamic boundary conditions into the theory.

9. Implications I – Knot Cobordisms

In this section we will give the construction of cobordism maps in knot Floer homologies for $\widehat{\mathrm{HFK}}$, $\mathrm{HFK}^{\bullet,-}$ and $\mathrm{HFK}^{\bullet,\infty}$ for simplicity, i.e. for theories in which the intersection condition $n_z = 0$ is imposed and n_w is arbitrary. In §9.5 we sketch the adaptions that have to be made for theories for which n_z is arbitrary. In the following, we will work with diagrams which are weakly admissible, where weak admissibility is defined with respect to the point z (see [4, Definition 4.10]). We will write $\mathrm{HFK}^{\bullet,\circ}$ to indicate, that we consider $\widehat{\mathrm{HFK}} = \mathrm{HFK}^{\bullet,\bullet}$ or $\mathrm{HFK}^{\bullet,-}$ (or even $\mathrm{HFK}^{\bullet,\infty}$).

We expect the reader to be familiar with the work [8] of Ozsváth and Szabó. The idea to associate to a cobordism W between two 3-manifolds Y and Y' a map between the associated Floer homologies is similar to the constructions in [8] and goes as follows: First observe that the cobordism W admits a handle decomposition relative to the boundary component Y with no 0-handles and no 4-handles (cf. [1]). To use the notation of [1], the boundary ∂W can be written as $\partial_+ W \sqcup \overline{\partial_- W}$, where one of these components might be empty. In our case, $\partial_- W = Y$ and $\partial_+ W = Y'$. Since both components are non-empty, we do not require 0-handles and 4-handles (see [1, Proposition 4.2.13]). Furthermore, we may think the handles to be attached in order of increasing index and the handles of the same index to be attached simultaneously (see [1, Proposition 4.2.7]). Thus, we may split up W as

$$W = W_1 \cup_\partial W_2 \cup_\partial W_3$$

where W_i is built by the handles of index i. To associate a map $F_W^{\bullet;\circ}$ to W we will choose a splitting of W into handles and then associate to W_i a map between the Floer homologies of the boundary components. Then, $F_W^{\bullet;\circ}$ will be the composition of these three maps. The maps defined, here, will be defined similarly as Oszváth and Szabó do it in their paper.

Definition 9.1. Let Y, Y' be closed, oriented 3-manifolds with knots $K \subset Y$ and $K' \subset Y'$. A cobordism W between (Y, K) and (Y', K') is a pair (N, ϕ) where N is a four-manifold with boundary $\partial N = -Y \sqcup Y'$ and ϕ is a proper embedding of the cylinder $[0,1] \times \mathbb{S}^1$ into N which maps its boundary to $K \sqcup K'$. We call W a **knot cobordism** from (Y, K) to (Y', K').

For example, such a cobordism is given by attaching a 2-handle $\mathrm{h}^{(4,2)}$ in the complement of the knot K. The cobordism $N = [0,1] \times Y \cup_\partial \mathrm{h}^{(4,2)}$ admits a canonical embedding ϕ of the cylinder into N, i.e. the embedding is given by the canonical inclusion

$$[0,1] \times K \hookrightarrow N$$

and we define $K' = \{1\} \times K$.

Definition 9.2. We say that two knot cobordisms W and W' are **equivalent** if there is a diffeomorphism $\psi\colon N \longrightarrow N'$ such that $\phi' = \psi \circ \phi$.

Definition 9.3. A handle decomposition of a knot cobordism $W = (N, \phi)$ between (Y, K) and (Y', K') is a handle decomposition $\mathcal{H}$ of N relative to Y using handles whose attaching spheres are all disjoint from K.

Observe that the embedding ϕ can be extended to an embedding ψ of $[0,1] \times \nu K$ into N which again is disjoint from the handles. Thus, a handle decomposition of a knot cobordism N can be defined equivalently as a handle decomposition of the manifold $\overline{N \backslash ([0,1] \times \nu K)}$ relative to $Y \backslash \nu K$, which we regard as a cobordism between $Y \backslash \nu K$ and $Y' \backslash \nu K'$. Observe, that our notion of knot cobordism is equivalent to the notion of *special cobordisms* of Juhász given in [2].

Theorem 9.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two handle decompositions of a knot cobordism N between (Y, K) and (Y', K') then we can transform $\mathcal{H}_1$ into $\mathcal{H}_2$ through handle decompositions of knot cobordisms using a finite sequence of handle slides, handle pair creation/annihilation and isotopies within levels.*

Proof. This follows from the observation that a knot cobordism is especially a special cobordism of Juhász and the results from [2]. □

9.1. **One-handles.** Suppose we are given a closed, oriented 3-manifold Y with knot K in it. Attach to the trivial cobordism $[0,1] \times Y$ a 4-dimensional 1-handle to the boundary $\{1\} \times Y$, where the attaching spheres of the 1-handles should be attached in the complement of K in Y. Denote the resulting cobordism by U. The boundary of the cobordism U is given as $\partial U = -Y \sqcup Y \# \mathbb{S}^2 \times \mathbb{S}^1$. Observe that $[0,1] \times K$ admits a natural embedding into U with $\{1\} \times K$ being mapped into $Y \#(\mathbb{S}^2 \times \mathbb{S}^1)$. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a Heegaard diagram adapted to the pair (Y, K) and let $(E, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, z_0)$ be a standard Heegaard diagram for $\mathbb{S}^2 \times \mathbb{S}^1$ (see [8, Definition 2.8]) so that the α-circle in $\boldsymbol{\alpha}_0$ and the β-circle in $\boldsymbol{\beta}_0$ meet in a single pair of intersection points. Denote by $\mathcal{E}$ the diagram $(\Sigma, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0, w_0, z_0)$ where w_0 is a point in $\Sigma \backslash (\boldsymbol{\alpha}_0 \cup \boldsymbol{\beta}_0)$ which lies in the same component as z_0. Furthermore, denote by θ the intersection point with higher relative grading. By [9, Corollary 6.8], we know that $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w, z') = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z) \# \mathcal{E}$ is a Heegaard diagram adapted to K and (the arguments given in Corollary 6.8 carry over verbatim for the HFK$^{\bullet,-}$-case)

$$(9.1) \qquad \begin{aligned} \mathrm{HFK}^{\bullet,\circ}(Y \# \mathbb{S}^2 \times \mathbb{S}^1, K) &\cong H_*(\mathrm{CFK}^{\bullet,\circ}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z) \otimes_{\mathbb{Z}_2[U]} \mathrm{CFK}^{\bullet,\circ}(\mathcal{E})) \\ &\cong \mathrm{HFK}^{\bullet,\circ}(Y, K) \otimes_{\mathbb{Z}_2[U]} \mathrm{HFK}^{\bullet,\circ}(\mathbb{S}^2 \times \mathbb{S}^1, U) \end{aligned}$$

Thus, we define a map

$$g_{U,\mathfrak{s}}^{\bullet,\circ}\colon \mathrm{CFK}^{\bullet,\circ}(\boldsymbol{\alpha}, \boldsymbol{\beta}; \mathfrak{s}) \longrightarrow \mathrm{CFK}^{\bullet,\circ}(\boldsymbol{\alpha}', \boldsymbol{\beta}'; \mathfrak{s} \# \mathfrak{s}_0)$$

by sending an element $U^i \cdot \boldsymbol{x}$, with $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, to $U^i \cdot \boldsymbol{x} \otimes \theta$. The moduli spaces in the definition of the differential on the right split as in the case of the $\widehat{\mathrm{HF}}$-theory (see [9, Corollary 6.8] for the hat-theory and [5, Theorem 1.5]) and, thus, the map is chain. Denote by $G_{U,\mathfrak{s}}^{\bullet,\circ}$ the induced map between the knot Floer homologies.

9.2. **Three-handles.** Let Y be a closed, oriented 3-manifold and $K \subset Y$ a knot in Y. Suppose V is a cobordism obtained by adding a single three-handle along a non-separating 2-sphere in Y which is disjoint from K. The boundary components of V are (Y, K) and (Y', K') with $Y' = Y \# \mathbb{S}^2 \times \mathbb{S}^1$ and K' sitting inside Y. We would like to remind the reader of [8, Lemma 4.11]: In this situation we can find a Heegaard diagram

$$\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z') = \mathcal{H} \# \mathcal{E}$$

of Y' where $\mathcal{E}$ is a standard Heegaard diagram for $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ a Heegaard diagram for Y. Furthermore, if two such split diagrams $\mathcal{H}'_i = \mathcal{H}_i \# \mathcal{E}$, for $i = 1, 2$, are equivalent, then $\mathcal{H}_1$ and $\mathcal{H}_2$ are equivalent.

It is easy to find $\mathcal{H}'$ which is adapted to the knot K' and still splits into $\mathcal{H}$ and $\mathcal{E}$. Furthermore, observe that $\mathcal{H}$ is a Heegaard diagram of Y adapted to K. Thus, we may proceed as done in [8]: We define a map

$$e^{\bullet, \circ}_{V, \mathfrak{s}} \colon \operatorname{CFK}^{\bullet, \circ}(\mathcal{H}', \mathfrak{s}|_{Y'}) \longrightarrow \operatorname{CFK}^{\bullet, \circ}(\mathcal{H}, \mathfrak{s}|_Y)$$

by sending an element $U^i \cdot \boldsymbol{x} \otimes \boldsymbol{y}$ with $\boldsymbol{x} \otimes \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}'} \cap \mathbb{T}_{\boldsymbol{\beta}'}$ to $U^i \cdot \boldsymbol{x}$ if $\boldsymbol{y}$ is the minimal intersection point of $\mathbb{S}^2 \times \mathbb{S}^1$ and to zero otherwise. As in the case of 1-handles, the moduli spaces of Whitney disks of $\operatorname{CFK}^{\bullet, \circ}(\mathcal{H}')$ split (see Corollary 6.8 of [9]) and, thus, $e^{\bullet, \circ}_{V, \mathfrak{s}}$ is a chain map. Hence, we get

$$E^{\bullet, \circ}_{V, \mathfrak{s}} \colon \operatorname{HFK}^{\bullet, \circ}(Y', K'; \mathfrak{s}|_{Y'}) \longrightarrow \operatorname{HFK}^{\bullet, \circ}(Y, K; \mathfrak{s}|_Y).$$

Observe that the definition of the maps $E^{\bullet, \circ}$ and $G^{\bullet, \circ}$ do not use the base point w.

9.3. **Two-handles.** Suppose we are given a closed, oriented 3-manifold Y and a knot $K \subset Y$. Furthermore, let $\mathbb{L}$ be a framed link in Y which is disjoint from K. We call such a link **admissible**. Analogous to the case of knots it is possible to find a Heegaard diagram **subordinate** to the link $K \sqcup \mathbb{L}$ (see [6] or [9, 10]). To describe such a diagram, let $\mathbb{L}_1, \ldots, \mathbb{L}_k$ be the components of $\mathbb{L}$. Then, there is a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ of Y such that K intersects β_1 once, transversely and is disjoint from the other $\boldsymbol{\beta}$-circles and for $i = 1, \ldots, k$ the knot $\mathbb{L}_i$ intersects β_{i+1}, transversely and is disjoint from the other $\boldsymbol{\beta}$-circles. The pair (w, z) determines the knot K in the usual way, i.e. in the way introduced by Ozsváth and Szabó in [6]. The diagram is characterized by the property that it comes from a handle decomposition (see [1] for a definition of relative handle decompositions) of $Y \backslash \nu(K \sqcup \mathbb{L})$ relative to $\partial(\nu(K \sqcup \mathbb{L}))$ (cf. [9, 10]). Using the Kirby calculus picture behind these subordinate diagrams as explained for instance in [9, 10], it is easy to see the following.

Proposition 9.5. *Let Y be a closed, oriented 3-manifold and $K \subset Y$ a knot. Let $\mathbb{L}$ be a framed link and $\mathcal{H}_i = (\Sigma_i, \boldsymbol{\alpha}_i, \boldsymbol{\beta}_i, w, z)$ be two diagrams subordinate to the link $K \sqcup \mathbb{L}$. Denote by I the interval inside K connecting z with w, interpreted as sitting in Σ. Then these diagrams are isomorphic after a finite sequence of the following moves:*

(m_1) *Handle slides and isotopies among the $\boldsymbol{\alpha}$-curves. These isotopies may not cross I.*

(m_2) *Handle slides and isotopies among the $\beta_{k+2}, \ldots, \beta_g$. These isotopies may not cross I.*

(m_3) *Handle slides of one of the β_i, $i \leq k+1$ over one of the β_j, $j \geq k+2$. These isotopies may not cross I.*

(m_4) *Stabilizations/destabilizations.*

This proposition can be proved with a straightforward adaption of the proof of [9, Lemma 2.3]. Given a subordinate diagram we may deduce a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, w, z)$ with the following properties: γ_1 and $\gamma_{k+2}, \dots, \gamma_g$ are small isotopic translates of β_1 and $\beta_{k+2}, \dots, \beta_g$. In fact, every pair β_i, γ_i meet in a canceling pair of intersection points. The curve γ_i, for $i = 2, \dots, k+1$, is determined by the framing of $\mathbb{L}_{i-1}$. Recall from [4, §8.1.] (or [8, §4.1.]) that the triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ determines a cobordism $X_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}}$ with three boundary components denoted by $Y_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$, $Y_{\boldsymbol{\beta}, \boldsymbol{\gamma}}$ and $Y_{\boldsymbol{\alpha}, \boldsymbol{\gamma}}$. For $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, $\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\mathfrak{s} \in \text{Spin}^c(X_{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}})$ we define

$$f_{\mathfrak{s}}^{\bullet, \circ}(\boldsymbol{x} \otimes \boldsymbol{y}) = \sum_{\boldsymbol{q} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\gamma}}, i \geq 0} \# \big(\mathcal{M}_{(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta}); \mathfrak{s}}^{0;i}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{q}) \big) \cdot U^i \, \boldsymbol{q},$$

where $\mathcal{M}_{(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{\beta}); \mathfrak{s}}^{0;i}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{q})$ denotes the moduli space of holomorphic Whitney triangles ϕ which connect $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{q}$ with boundary conditions in $\mathbb{T}_{\boldsymbol{\alpha}}$, $\mathbb{T}_{\boldsymbol{\gamma}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ such that $n_w(\phi) = i$ and $\mathfrak{s}_z(\phi) = \mathfrak{s}$. We extend this to a bilinear pairing

$$f_{\mathfrak{s}}^{\bullet, \circ} \colon \text{CFK}^{\bullet, \circ}(\boldsymbol{\alpha}, \boldsymbol{\beta}, w, z; \mathfrak{s}_{\boldsymbol{\alpha}\boldsymbol{\beta}}) \otimes \text{CFK}^{\bullet, \circ}(\boldsymbol{\beta}, \boldsymbol{\gamma}, w, z; \mathfrak{s}_{\boldsymbol{\beta}\boldsymbol{\gamma}}) \longrightarrow \text{CFK}^{\bullet, \circ}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, w, z; \mathfrak{s}_{\boldsymbol{\alpha}\boldsymbol{\gamma}})$$

where $\mathfrak{s}_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \mathfrak{s}|_{Y_{\boldsymbol{\alpha}, \boldsymbol{\beta}}}$, $\mathfrak{s}_{\boldsymbol{\beta}\boldsymbol{\gamma}} = \mathfrak{s}|_{Y_{\boldsymbol{\beta}, \boldsymbol{\gamma}}}$ and $\mathfrak{s}_{\boldsymbol{\alpha}\boldsymbol{\gamma}} = \mathfrak{s}|_{Y_{\boldsymbol{\alpha}, \boldsymbol{\gamma}}}$.

Remark 1. Our notation of the boundary conditions at the edges of the triangle differs from the notation introduced by Ozsváth and Szabó. We adapted the notation to the conventions introduced at the beginning of §3.1.

Lemma 9.6. *The Heegaard diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, w, z)$ is subordinate to the unknot U in the manifold $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$.*

Proof. The diagram clearly represents $\#^k(\mathbb{S}^2 \times \mathbb{S}^1)$ (see [4, 5]). By definition, the points w and z can be connected in Σ in the complement of the β-curves. Let c be such a curve. We claim that we can choose the curve c to sit in the complement of the $\boldsymbol{\gamma}$-curves. If we are able to show this, we are done, as we can use a small push-off c' of c and connect z with w in the complement of the $\boldsymbol{\gamma}$-curves. By definition, the union $c' \cup c$ is isotopic to the knot represented by the pair (w, z). The curve $c' \cup c$ is contractible and, thus, (w, z) represents the unknot.

The curve γ_1 is a small isotopic translate of the curve β_1. Since c and β_1 are disjoint, the curves c and γ_1 are disjoint, too. The curves β_i, $i = 2, \dots, k+1$ are meridians of torus components of the surface Σ. The curve $\boldsymbol{\gamma}_i$ is isotopic to $n_i \beta_i + \lambda_i$ where n_i is a suitable integer and λ_i a longitude of the corresponding torus component associated to β_i. Hence, the γ_i can be thought of as staying outside of the torus component in which c lies in. Finally, there are the curves γ_j, $j \geq k+2$. The curve γ_j is an isotopic translate of the curve β_j which is disjoint from c. Hence, γ_j can be thought of as being disjoint from c. $\square$

Consequently, we have that $\text{HFK}^{\bullet, \circ}(\boldsymbol{\beta}, \boldsymbol{\gamma}, w, z)$ admits a top-dimensional generator, $\widehat{\Theta}$ say (see [8]). Denote by Y' the manifold obtained by the surgery along the framed link $\mathbb{L}$ and

denote by K' the knot K after the surgery. Since $Y_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cong Y$ and $Y_{\boldsymbol{\alpha},\boldsymbol{\gamma}} \cong Y'$, it is possible to define

$$F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ} \colon \operatorname{HFK}^{\circ}(Y, K; \mathfrak{s}|_Y) \longrightarrow \operatorname{HFK}^{\circ}(Y', K'; \mathfrak{s}|_{Y'})$$

as the map induced by $f_{\mathfrak{s}}^{\bullet,\circ}(\,\cdot\,, \widehat{\Theta})$ in homology.

9.4. **Invariants of Cobordisms.** Given a knot cobordism W from (Y, K) to (Y', K'), we choose a handle decomposition of it, i.e. we choose a splitting

$$W = W_1 \cup_\partial W_2 \cup_\partial W_3$$

where W_i is obtained by attaching i-handles. Let $\mathbb{L}$ be the framed link associated to the 2-handle attachments in W_2, then we define

$$F_{W;\mathfrak{s}}^{\bullet,\circ} = E_{W_3;\mathfrak{s}}^{\bullet,\circ} \circ F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ} \circ G_{W_1;\mathfrak{s}}^{\bullet,\circ}.$$

Proof of Theorem 1.2. This theorem has to be proved by showing that the map $F_{W;\mathfrak{s}}^{\bullet,\circ}$ does not depend on the handle decomposition of W and the data associated to it. It is easy to observe that on the chain level we have that

$$(9.2) \qquad \begin{aligned} G_{U;\mathfrak{s}}^{\bullet,\circ} &= G_{U;\mathfrak{s}}^{\circ} \\ E_{V;\mathfrak{s}}^{\bullet,\circ} &= E_{V;\mathfrak{s}}^{\circ}, \end{aligned}$$

where $G_{U;\mathfrak{s}}^{\circ}$ and $E_{V;\mathfrak{s}}^{\circ}$ are the maps associated to 1-handles and 3-handles, respectively, which were defined by Ozsváth and Szabó in [8]. Thus, the maps on the left of (9.2) and on the right of (9.2) basically have the same properties. Examining the work [8], we see that to make the invariance work in the knot Floer case there is one property that is central: We have to prove that the map $F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ}$ is independent of the choice of subordinate Heegaard diagram. In [8], Ozsváth and Szabó prove this for the maps $F_{\mathbb{L};\mathfrak{s}}^{\circ}$ by showing that they *commute* with all maps induced by admissible Heegaard moves. We will do this for the maps $F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ}$ in the sequel: Given two Heegaard diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', w, z)$ subordinate to the link $\mathbb{L}$, there is a sequence of subordinate Heegaard diagrams $\mathcal{H}_1, \ldots, \mathcal{H}_n$ with $\mathcal{H} = \mathcal{H}_1$ and $\mathcal{H}' = \mathcal{H}_n$ such that we get from $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$ by one of the moves introduced in Proposition 9.5. Each of these moves induces an isomorphism

$$\Psi_{\mathfrak{s};i}^{\bullet,\circ} \colon \operatorname{HFK}^{\bullet,\circ}(\mathcal{H}_i; \mathfrak{s}) \longrightarrow \operatorname{HFK}^{\bullet,\circ}(\mathcal{H}_{i+1}; \mathfrak{s})$$

between the respective homologies. We have to prove that

$$(9.3) \qquad \Psi_{\mathfrak{s};i}^{\bullet,\circ} \circ F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ} + F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ} \circ \Psi_{\mathfrak{s};i}^{\bullet,\circ} = 0.$$

If $\Psi_{\mathfrak{s};i}^{\bullet,\circ}$ is induced by a stabilization/destabilization, the proof from the Heegaard Floer case, i.e. the proof of [8, Lemma 4.7], carries over verbatim. Otherwise, we proceed as follows: We first start defining

$$F_{\mathbb{L}}^{\bullet,\circ} = \sum_{\mathfrak{s}} F_{\mathbb{L};\mathfrak{s}}^{\bullet,\circ}$$

and, correspondingly, we define $\Psi_i^{\bullet,\circ}$. Observe, that the sum is finite due to the fact that we demand the condition $n_z = 0$ and use diagrams that are weakly admissible with respect

to the point z. We know that, forgetting the point w, we obtain isomorphisms Ψ_i and maps $\widehat{F}_{\mathbb{L};\mathfrak{s}}$ between the associated Heegaard Floer theories which fulfill the equation

$$(9.4) \qquad \widehat{\Psi}_i \circ \widehat{F}_{\mathbb{L}} + \widehat{F}_{\mathbb{L}} \circ \widehat{\Psi}_i = 0$$

as shown by Ozsváth and Szabó. They derived this equation by counting ends of a suitable 1-dimensional moduli space. Hence, their proof has a formulation in terms of the symbol homology theory: As shown in the previous sections, there is a symbol homology theory $\mathfrak{sh}_*$ (see §8.1 and §8.2) such that both $\widehat{\Psi}_i$ and $\widehat{F}_{\mathbb{L}}$ admit canonical symbols in this theory, $\mathfrak{s}_\Psi$ and $\mathfrak{s}_{\widehat{F}}$ say. Ozsváth and Szabó's proof can be interpreted in the language of symbol homology which gives $\mathfrak{s}_{\widehat{F}} \boxtimes \mathfrak{s}_\Psi \boxplus \mathfrak{s}_{\widehat{F}} \boxtimes \mathfrak{s}_\Psi = 0$ in $\mathfrak{sh}_*$. Since the Heegaard move underlying the map Ψ_i respects the point w, we know that there are filtering morphisms

$$\mathfrak{F}_* \colon \mathfrak{sh}_* \longrightarrow \mathfrak{sh}_*^w$$
$$\mathfrak{F}_{U;*} \colon \mathfrak{sh}_* \longrightarrow \mathfrak{sh}_{U,*}$$

such that Theorem 4.3 holds (alternatively, Theorem 1.1 and Theorem 7.2). Since $\mathfrak{s}_{\widehat{F}}$ fulfills the property P with $P(X) = X \boxtimes \mathfrak{s}_\Psi \boxplus X \boxtimes \mathfrak{s}_\Psi$, the filtered symbols $\mathfrak{F}_*(\mathfrak{s}_{\widehat{F}})$ and $\mathfrak{F}_{U;*}(\mathfrak{s}_{\widehat{F}})$ fulfill the properties $P_{\mathfrak{F}}$ and $P_{\mathfrak{F}_U}$, respectively. But, by construction, $\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{\widehat{F}})) = F_{\mathbb{L};\mathfrak{s}}^{\bullet,\bullet}$ and $\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_\Psi)) = \Psi_i^{\bullet,\bullet}$. So, property $P_{\mathfrak{F}}$ implies

$$\begin{aligned}
0 = \mathfrak{ev}_*\big(P_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s}_{\widehat{F}}))\big) &= \mathfrak{ev}_*\big(\mathfrak{F}_*(\mathfrak{s}_{\widehat{F}}) \boxtimes \mathfrak{F}_*(\mathfrak{s}_\Psi) \boxplus \mathfrak{F}_*(\mathfrak{s}_{\widehat{F}}) \boxtimes \mathfrak{F}_*(\mathfrak{s}_\Psi)\big) \\
&= \mathfrak{ev}_*\big(\mathfrak{F}_*(\mathfrak{s}_\Psi)\big) \circ \mathfrak{ev}_*\big(\mathfrak{F}_*(\mathfrak{s}_{\widehat{F}})\big) + \mathfrak{ev}_*\big(\mathfrak{F}_*(\mathfrak{s}_\Psi)\big) \circ \mathfrak{ev}_*\big(\mathfrak{F}_*(\mathfrak{s}_{\widehat{F}})\big) \\
&= \Psi_i^{\bullet,\bullet} \circ F_{\mathbb{L}}^{\bullet,\bullet} + F_{\mathbb{L}}^{\bullet,\bullet} \circ \Psi_i^{\bullet,\bullet}
\end{aligned}$$

and, correspondingly, property $P_{\mathfrak{F}_U}$ implies

$$0 = \mathfrak{ev}_*(P_{\mathfrak{F}_U}(\mathfrak{F}_{U;*}(\mathfrak{s}_{\widehat{F}}))) = \Psi_i^{\bullet,-} \circ F_{\mathbb{L}}^{\bullet,-} + F_{\mathbb{L}}^{\bullet,-} \circ \Psi_i^{\bullet,-}.$$

Hence, the map $F_{\mathbb{L}}^{\bullet,\circ}$ is an invariant as stated.

To get the refined statements, i.e. equation (9.3), we make the following adaptions: Observe, that to encode the maps Ψ_i and $F_{\mathbb{L}}^{\bullet,\circ}$ as symbols, we need a set B which contains at most four sets of attaching circles. Hence, in the corresponding symbol algebra only bigons, triangles and rectangles appear. For all of these n-gons, Ozsváth and Szabó introduced the notion of associated Spin^c-structure (see [4, Proposition 8.5 and §8.1.5]). We alter the theory $\mathfrak{sh}_*$ by attaching an additional datum to the generators: We decorate moduli spaces as done in §3 and, additionally, attach a Spin^c-structure. A choice of Spin^c-structure on a Whitney polygon especially induces a choice of Spin^c-structures on its vertices. We follow the lines from §3 verbatim except for two issues: First, when defining the $\boxtimes$-product of pre-generators $\mathcal{A}_1$ and $\mathcal{A}_2$ with Spin^c-structures we bring in the chosen Spin^c-structure by saying that the data of two elements match, if the data of the factors without Spin^c-structures match (in the sense given in §3.1) and if the Spin^c-structure at their common vertex coincides (see Definition 3.3). Second, when defining the differential $\partial_{\mathfrak{sh}}$ of a generator $\mathcal{A}$ with Spin^c-structure $\mathfrak{s}$, we bring in the additional datum in the following way: Recall, that $\partial_{\mathfrak{sh}}$ is modeled on $\partial^{\mathrm{codim}\,1}(\pi(\mathcal{A}))$. For each component $A_1^{ab;\boldsymbol{q}} \times A_2^{ab;\boldsymbol{q}}$ of $\partial^{\mathrm{codim}\,1}(\pi(\mathcal{A}))$

(cf. §3.2) the Spin^c-structure $\mathfrak{s}$ of $\mathcal{A}$ induces Spin^c-structures $\mathfrak{s}_i$ on $A_i^{ab;\boldsymbol{q}}$, for $i = 1, 2$. We will attach $\mathfrak{s}_i$ to the corresponding $\mathcal{A}_i^{ab;\boldsymbol{q}}$ (cf. §3.2) and then proceed as in §3.2. The resulting homology theory shall be denoted by $\mathfrak{sh}_*^c$. On this homology theory there exists a map

$$\mathfrak{ev}^c \colon \mathfrak{sh}_*^c \longrightarrow \widehat{\mathbb{MOR}}$$

which is defined as $\mathfrak{ev}$: Using the notation from Proposition 3.13, for a generator with Spin^c-structure $\mathfrak{s}$, i.e. $\mathcal{A} = A_{(P^\bullet, F^\downarrow, F^\uparrow);\mathfrak{s}}$, we define

$$\mathfrak{ev}^c(\mathcal{A})(\boldsymbol{x}) = \#_\mathfrak{s}\big(A(\boldsymbol{x}, \boldsymbol{q}, \boldsymbol{r})\big) \cdot \boldsymbol{r}$$

where $\#_\mathfrak{s}$ only counts elements whose Spin^c-structure equal to $\mathfrak{s}$. We extend to the symbol algebra as done in (3.12). This theory also comes with variants $\mathfrak{sh}_*^{w;c}$ and $\mathfrak{sh}_{U,*}^c$ and those defined in §4 and §7. We start with the equation

$$\widehat{\Psi}_{\mathfrak{s};i} \circ \widehat{F}_{\mathbb{L};\mathfrak{s}} + \widehat{F}_{\mathbb{L};\mathfrak{s}} \circ \widehat{\Psi}_{\mathfrak{s};i} = 0,$$

which was proved by Ozsváth and Szabó and copy the arguments from above and apply Theorem 7.2. This will provide us with equality (9.3). $\qquad\square$

9.5. Other Knot Floer Theories.

The construction of the maps presented above with the obvious notational adaptions – i.e. swapping the roles of w and z – also provide cobordism maps in the theories HFK$^{\circ,\bullet}$. To apply the symbol homology for the invariance proof we have to alter the theory a little. In the proof of Theorem 1.2 the invariances of the maps were transferred from the $\widehat{\mathrm{HF}}$-theory using the filtering maps $\mathfrak{F}$ and $\mathfrak{F}_U$. In HFK$^{\circ,\bullet}$, invariances cannot be related to the $\widehat{\mathrm{HF}}$-theory but to the HF$^\circ$-theory. We have to define a symbol homology that captures the theory HF$^\circ$ (in the sense specified in the introduction). This is done by decorating moduli spaces of Whitney polygons for which both n_z and n_w are arbitrary. We also attach a choice of Spin^c-structure to the moduli spaces (as done in the proof of Theorem 1.2). The construction then follows the lines of §3. Because of the Spin^c-structures in the construction we will use $\mathfrak{ev}^c$ to interpret the elements of the associated symbol algebra as maps. As filtering morphism we will need one of type $\mathfrak{F}$, i.e. one analogous as the one defined in §4.1. Then, the invariance proof will proceed exactly as the proof of Theorem 1.2.

10. Implication II – A Surgery Exact Triangle

In §9 we constructed cobordism maps for knot Floer homologies and used the techniques from symbol homologies in their invariance proof. Here, we would like to present other examples of how the techniques can be applied. The fact that the knot Floer homologies are invariants of a pair (Y, K) is well-known, however the proof of Corollary 10.1 shows that these invariances are in some way *inherited* from the invariances of the $\widehat{\mathrm{HF}}$-theory. Furthermore, we prove a generalization of the surgery exact sequence presented in [6, Theorem 8.2]. Again, the proof rests on the symbol homology approach we discussed.

Corollary 10.1 (see [6]). *The knot Floer homologies* HFK$^{\bullet,\circ}(Y, K)$ *are invariants of the pair* (Y, K) *because* $\widehat{\mathrm{HF}}(Y)$ *is an invariant of* Y.

Proof. Suppose we are given two Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ subordinate to a pair (Y, K). We have to see that the Floer homologies associated to these diagrams are isomorphic. There is a sequence $\mathcal{H}_1, \ldots, \mathcal{H}_n$ of Heegaard diagrams subordinate to (Y, K) such that we transform $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$ by using one of the moves given in Proposition 9.5. The only non-trivial statements to prove are independence of the choice of almost complex structure and invariance under isotopies of the attaching circles. The arguments Ozsváth and Szabó gave for stabilizations and handle slides in the Heegaard Floer case are completely independent of the introduction of an additional base point w. Thus, these proofs immediately carry over to the knot Floer case. We will proceed to give the arguments for an isotopy I_t of the $\boldsymbol{\beta}$-circles. The other cases can be proved in the same fashion. Thus, suppose we obtained $\mathcal{H}_{i+1} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}')$ from $\mathcal{H}_i = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ by the isotopy I_t. The isotopy I_t and its inverse I_{1-t} induce maps

$$\widehat{\Psi}_{to} : \widehat{\mathrm{CF}}(\mathcal{H}) \longrightarrow \widehat{\mathrm{CF}}(\mathcal{H}')$$
$$\widehat{\Psi}_{ot} : \widehat{\mathrm{CF}}(\mathcal{H}') \longrightarrow \widehat{\mathrm{CF}}(\mathcal{H}),$$

which are chain maps and their composition $\widehat{\Psi}_{ot} \circ \widehat{\Psi}_{to}$ is chain homotopic to the identity. Denote by $\mathfrak{s}_{\mathcal{H}}$ and $\mathfrak{s}_{\mathcal{H}'}$ the canonical symbols of the differentials $\widehat{\partial}_{\mathcal{H}}$ and $\widehat{\partial}_{\mathcal{H}'}$. Furthermore, denote by H the chain homotopy and by $\mathfrak{s}_H$ the associated canonical symbol (in the perturbed symbol homology). Finally, denote by $\mathfrak{s}_{to}$ and $\mathfrak{s}_{ot}$ the canonical symbol of $\widehat{\Psi}_{to}$ and $\widehat{\Psi}_{ot}$. Using the symbol homologies we get

$$\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to})) : \mathrm{CFK}^{\bullet,\bullet}(\mathcal{H}) \longrightarrow \mathrm{CFK}^{\bullet,\bullet}(\mathcal{H}')$$
$$\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{ot})) : \mathrm{CFK}^{\bullet,\bullet}(\mathcal{H}') \longrightarrow \mathrm{CFK}^{\bullet,\bullet}(\mathcal{H})$$
$$\mathfrak{ev}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{to})) : \mathrm{CFK}^{\bullet,-}(\mathcal{H}) \longrightarrow \mathrm{CFK}^{\bullet,-}(\mathcal{H}')$$
$$\mathfrak{ev}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{ot})) : \mathrm{CFK}^{\bullet,-}(\mathcal{H}') \longrightarrow \mathrm{CFK}^{\bullet,-}(\mathcal{H}).$$

We want to prove that the first map is a chain map and that the composition of the upper two is chain homotopic to the identity: The fact that $\widehat{\Psi}_{to}$ is a chain map is given by the property P with

$$P(X) = X \boxtimes \mathfrak{s}_{\mathcal{H}'} \boxplus \mathfrak{s}_{\mathcal{H}} \boxtimes X.$$

The fact that H is a chain homotopy to the identity is encoded by Q with

$$Q(X) = \mathfrak{s}_{to} \boxtimes \mathfrak{s}_{ot} \boxplus \mathbb{O} \boxplus X \boxtimes \mathfrak{s}_{\mathcal{H}'} \boxplus \mathfrak{s}_{\mathcal{H}} \boxtimes X.$$

The invariance proof of Ozsváth and Szabó shows that $P(\mathfrak{s}_{to}) = 0$ and that $Q(\mathfrak{s}_H) = 0$. By Theorem 4.3 the symbol $\mathfrak{F}_*(\mathfrak{s}_{to})$ fulfills property $P_{\mathfrak{F}}$ and the symbol $\mathfrak{F}_*(\mathfrak{s}_H)$ fulfills the property $Q_{\mathfrak{F}}$. Thus, both $P_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s}_{to}))$ and $Q_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s}_H))$ vanish. Applying the morphism $\mathfrak{ev}_*$ we see that

$$0 = \mathfrak{ev}_*(P_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s}_{to}))) = \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to})) \circ \partial_{\mathcal{H}}^{\bullet,\bullet} + \partial_{\mathcal{H}'}^{\bullet,\bullet} \circ \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to}))$$

holds and, correspondingly, that

$$\begin{aligned}
0 &= \mathfrak{ev}_*(Q_{\mathfrak{F}}(\mathfrak{F}_*(\mathfrak{s}_H))) \\
&= \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{ot})) \circ \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to})) + \mathrm{id} + \partial_{\mathcal{H}'}^{\bullet,\bullet} \circ \mathfrak{ev}_*(\mathfrak{F}(\mathfrak{s}_H)) + \mathfrak{ev}_*(\mathfrak{F}(\mathfrak{s}_H)) \circ \partial_{\mathcal{H}}^{\bullet,\bullet}.
\end{aligned}$$

Hence, $\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to}))$ is a chain map and the composition $\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{ot})) \circ \mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to}))$ chain homotopic to the identity. Similarly, we prove that $\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{ot}))$ is chain and that $\mathfrak{ev}_*(\mathfrak{F}_*(\mathfrak{s}_{to})) \circ$

$\mathfrak{cv}_*(\mathfrak{F}_*(\mathfrak{s}_{ot}))$ is chain homotopic to the identity. Using the filtering map $\mathfrak{F}_{U;*}$, we can prove the corresponding statements for $\mathfrak{cv}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{to}))$, $\mathfrak{cv}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{ot}))$, $\mathfrak{cv}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{to})) \circ \mathfrak{cv}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{ot}))$ and $\mathfrak{cv}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{ot})) \circ \mathfrak{cv}_*(\mathfrak{F}_{U;*}(\mathfrak{s}_{to}))$. $\qquad\square$

Now suppose we are given a closed, oriented 3-manifold Y and a knot $K \subset Y$. Given a knot $L \subset Y$ disjoint from K with framing n, we define (Y_n, K') as the pair we obtain by performing surgery along L. Denote by W_1 the induced knot cobordism. Then, we denote by (Y_{n+1}, K'') the pair we obtain from (Y_n, K') by performing a (-1)-surgery along a meridian L, μ say, and we denote by W_2 the associated knot cobordism. Finally, denote by W_3 the knot cobordism obtained by performing a (-1)-surgery along a meridian of μ.

Theorem 10.2. *In the situation defined above, the following sequence is exact.*

$$\mathrm{HFK}^{\bullet,\circ}(Y,K) \xrightarrow{F_{W_1}^{\bullet,\circ}} \mathrm{HFK}^{\bullet,\circ}(Y_n, K') \xrightarrow{F_{W_2}^{\bullet,\circ}} \mathrm{HFK}^{\bullet,\circ}(Y_{n+1}, K'')$$
$$\underleftarrow{\phantom{\mathrm{HFK}^{\bullet,\circ}(Y,K)}\quad F_{W_3}^{\bullet,\circ}\quad\phantom{\mathrm{HFK}}}$$

Proof. We use the mapping cone proof approach of Ozsváth and Szabó from [7]. They use an algebraic trick using mapping cones to prove exactness of the sequence, namely [7, Lemma 4.2]. To apply this lemma, they need to prove two properties, where the one is an associativity property of cobordism maps and the other a chain homotopy relation of cobordism maps. Both of these properties can be encoded as a property P in the corresponding symbol homologies. Hence, by Theorem 1.1 and Theorem 7.2 the corresponding associativity property and chain homotopy relation also hold in the knot Floer case which allows us to apply their Lemma 4.2 to get exactness. $\qquad\square$

In a similar vein, other properties and statements about cobordism maps can be easily transferred. Since the strategy of the proofs is always the same, we will leave this to the interested reader.

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